

# An improved time-dependent Hartree–Fock approach for scalar $\phi^4$ QFT

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## Abstract

The  $\lambda\phi^4$  model in a finite volume is studied within a non-gaussian Hartree–Fock approximation (tdHF) both at equilibrium and out of equilibrium, with particular attention to the structure of the ground state and of certain dynamical features in the broken symmetry phase. The mean-field coupled time-dependent Schrödinger equations for the modes of the scalar field are derived and the suitable procedure to renormalize them is outlined. A further controlled gaussian approximation of our tdHF approach is used in order to study the dynamical evolution of the system from non-equilibrium initial conditions characterized by a uniform condensate. We find that, during the slow rolling down, the long-wavelength quantum fluctuations do not grow to a macroscopic size but do scale with the linear size of the system, in accordance with similar results valid for the large  $N$  approximation of the  $O(N)$  model. This behavior undermines in a precise way the gaussian approximation within our tdHF approach, which therefore appears as a viable mean to correct an unlikely feature of the standard HF factorization scheme, such as the so-called “stopping at the spinodal line” of the quantum fluctuations. We also study the dynamics of the system in infinite volume with particular attention to the asymptotic evolution in the broken symmetry phase. We are able to show that the fixed points of the evolution cover at most the classically metastable part of the static effective potential.

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## I. INTRODUCTION

A great effort has been devoted in the last few years in order to develop a deeper qualitative and quantitative understanding of systems described by interacting quantum fields out of equilibrium. There is a class of physical problems that requires the consistent treatment of time dependent mean-fields in interaction with their own quantum or thermal fluctuations. We may mention, among others, the problem of reheating of the universe after the inflationary era of exponential growth and cooling, and the time evolution of the scalar order parameter through the chiral phase transition, soon to be probed in the forthcoming heavy-ion experiments at CERN–SPS, BNL–RHIC and CERN–LHC. In these situations, a detailed description of the time-dependent dynamics is necessary to calculate the non-equilibrium properties of the system. Indeed, the development of practical general techniques and the advent of faster and cheaper computers have made possible the discovery of novel and unexpected phenomena, ranging from dissipative processes via particle production to novel aspects of symmetry breaking [1–4].

From the technical point of view, it should be pointed out, first of all, that a perturbative treatment of this dynamical problem is meaningful only when the early time evolution is considered. The presence of parametric resonant bands or spinodal instabilities (in the case, respectively, of unbroken or spontaneously broken symmetries) rapidly turns the dynamics completely non-linear and non-perturbative. Thus, the asymptotic evolution at late time can be consistently studied only if approximate non-perturbative methods are applied to the problem [1].

Quite recently one of these schemes, namely the large  $N$  expansion at leading order (LN) [5,6], has been used in order to clarify some dynamical aspects of the  $\phi^4$  theory in 3 spatial dimensions, reaching the conclusion that the non-perturbative and non-linear evolution of the system might eventually produce the onset of a form of non-equilibrium Bose–Einstein condensation (BEC) of the long-wavelength Goldstone bosons usually present in the broken symmetry phase [3,4,7]. Another very interesting result in [7] concerns the dynamical Maxwell construction, which reproduces the flat region of the effective potential in case of broken symmetry as asymptotic fixed points of the background evolution.

In a companion work [8] we have addressed the question of whether a standard BEC could actually take place as time goes on, by putting the system in a finite volume (a periodic box of size  $L$ ) and carefully studying the volume dependence of out-of-equilibrium features in the broken symmetry phase. We summarize here the main result contained in [8]. The numerical solution shows the presence of a time scale  $\tau_L$ , proportional to the linear size  $L$  of the system, at which finite volume effects start to manifest, with the remarkable consequence that the zero-mode quantum fluctuations cannot grow macroscopically large if they start with microscopic initial conditions. In fact, the size of low-lying widths at time  $\tau_L$  is of order  $L$ , to be compared to order  $L^{3/2}$  for the case of standard BEC. In other words we confirmed that the linear growth of the zero mode width, as found also by the authors of [3,4,7], really signals the onset of a novel form of dynamical BEC, quite different from the standard one described by equilibrium finite-temperature field theory. This interpretation is reinforced by the characteristics of the long-wavelength fluctuations' spectrum.

Since after all the large  $N$  approximation is equivalent to a Gaussian ansatz for the time-dependent density matrix of the system [2,9], one might still envisage a scenario in

which, while gaussian fluctuations would stay microscopic, non-gaussian fluctuations would grow in time to a macroscopic size, leading to an occupation number for the zero mode proportional to the volume  $L^3$  of the system. Therefore, in order to go beyond the gaussian approximation, we will consider in this work a time-dependent HF approach capable in principle of describing the dynamics of some non-gaussian fluctuations of a single scalar field with  $\phi^4$  interaction.

Before going into the details of the analysis, let us briefly summarize the main limitations and the most remarkable results of the study of a scalar field out of equilibrium within the gaussian HF scheme [1,10–12]. First of all, this scheme has the advantage of going beyond perturbation theory, in the sense that the (numerical) solution of the evolution equations will contain arbitrary powers of the coupling constant, corresponding to a non-trivial resummation of the perturbative series. For this reason, the method is able to take into account the quantum back-reaction on the fluctuations themselves, which shuts off their early exponential growth. This is achieved by the standard HF factorization of the quartic interaction, yielding a *time dependent* self-consistently determined mass term, which stabilizes the modes perturbatively unstable. The detailed numerical solution of the resulting dynamical equations clearly shows the dissipation associated with particle production, as a result of either parametric amplification in case of unbroken symmetry or spinodal instabilities in case of broken symmetry, as well as the shut off mechanism outlined above.

However, the standard HF method is really not controllable in the case of a single scalar field, while it becomes exact only in the  $N \rightarrow \infty$  limit. Moreover, previous approaches to the dynamics in this approximation scheme had the unlikely feature of maintaining a weak (logarithmic) cut-off dependence on the renormalized equations of motion of the order parameter and the mode functions [1].

In this article we consider the case of a single scalar field (i.e.  $N = 1$ ). With the aim of studying the dynamics of the model with the inclusion of some non-gaussian contributions, we introduce an improved time-dependent Hartree–Fock approach. Even if it is still based on a factorized trial wavefunction(al), it has the merit to keep the quartic interaction diagonal in momentum space, explicitly in the hamiltonians governing the evolution of each mode of the field. In this framework, issues like the static spontaneous symmetry breaking can be better understood, and the further gaussian approximation needed to study the dynamics can be better controlled. In particular, questions like out-of-equilibrium “quantum phase ordering” and “dynamical Bose–Einstein condensation” can be properly posed and answered within a verifiable approximation.

We also perform a detailed study of the asymptotic dynamics in infinite volume, with the aim of clarifying the issue of Maxwell construction in this approximation scheme. In fact, in the  $O(N)\Phi^4$  model at leading order, the asymptotic dynamical evolution of the mean field completely covers the spinodal region of the classical potential, which coincides with the flatness region of the effective potential. This is what is called *dynamical Maxwell construction* [7]. When we use the HF approximation for the case of  $N = 1$ , we find that the spinodal region and the flatness region are different and the question arise of whether a full or partial dynamical Maxwell construction still takes place.

In section II we set up the model in finite volume, defining all the relevant notations and the quantum representation we will be using to study the evolution of the system.

We introduce in section III our improved time-dependent Hartree–Fock (tdHF) ap-

proximation, which generalizes the standard gaussian self-consistent approach [13] to non-gaussian wave-functionals; we then derive the mean-field coupled time-dependent Schroedinger equations for the modes of the scalar field, under the assumption of a uniform condensate, see eqs (3.5), (3.6) and (3.7). A significant difference with respect to previous tdHF approaches [1] concerns the renormalization of ultraviolet divergences. In fact, by means of a single substitution of the bare coupling constant  $\lambda_b$  with the renormalized one  $\lambda$  in the Hartree-Fock hamiltonian, we obtain cut-off independent equations (apart from corrections in inverse powers, which are there due to the Landau pole). The substitution is introduced by hand, but is justified by simple diagrammatic considerations.

One advantage of not restricting a priori the self-consistent HF approximation to gaussian wave-functionals, is in the possibility of a better description of the vacuum structure in case of broken symmetry. In fact we can show quite explicitly that, in any finite volume, in the ground state the zero-mode of  $\phi$  field is concentrated around the two vacua of the broken symmetry, driving the probability distribution for any sufficiently wide smearing of the field into a two peaks shape. This is indeed what one would intuitively expect in case of symmetry breaking. On the other hand none of this appears in a dynamical evolution that starts from a distribution localized around a single value of the field in the spinodal region, confirming what already seen in the large  $N$  approach [8]. More precisely, within a further controlled gaussian approximation of our tdHF approach, one observes that initially microscopic quantum fluctuations never becomes macroscopic, suggesting that also non-gaussian fluctuations cannot reach macroscopic sizes. As a simple confirmation of this fact, consider the completely symmetric initial conditions  $\langle \phi \rangle = \langle \dot{\phi} \rangle = 0$  for the background: in this case we find that the dynamical equations for initially gaussian field fluctuations are identical to those of large  $N$  (apart for a rescaling of the coupling constant by a factor of three; cfr. ref. [8]), so that we observe the same asymptotic vanishing of the effective mass. However, this time no interpretation in terms of Goldstone theorem is possible, since the broken symmetry is discrete; rather, if the width of the zero-mode were allowed to evolve into a macroscopic size, then the effective mass would tend to a positive value, since the mass in case of discrete symmetry breaking is indeed larger than zero.

Anyway, also in the gaussian HF approach, we do find a whole class of cases which exhibit the time scale  $\tau_L$ . At that time, finite volume effects start to manifest and the size of the low-lying widths is of order  $L$ . We then discuss why this undermines the self-consistency of the gaussian approximation, imposing the need of further study, both analytical and numerical.

In section IV we study the asymptotic evolution in the broken symmetry phase, in infinite volume, when the expectation value starts within the region between the two minima of the potential. We are able to show by precise numerical simulations, that the fixed points of the background evolution do not cover the static flat region completely. On the contrary, the spinodal region seems to be absolutely forbidden for the late time values of the mean field. Thus, as far as the asymptotic evolution is concerned, our numerical results lead to the following conclusions. We can distinguish the points lying between the two minima in a fashion reminiscent of the static classification: first, the values satisfying the property  $v/\sqrt{3} < |\bar{\phi}_\infty| \leq v$  are metastable points, in the sense that they are fixed points of the background evolution, no matter which initial condition comprised in the interval  $(-v, v)$  we choose for the expectation value  $\bar{\phi}$ ; secondly, the points included in the interval

$0 < |\bar{\phi}_\infty| < v/\sqrt{3}$  are unstable points, because if the mean field starts from one of them, after an early slow rolling down, it starts to oscillate with decreasing amplitude around a point inside the classical metastable interval. Obviously,  $\bar{\phi} = v$  is the point of stable equilibrium, and  $\bar{\phi} = 0$  is a point of unstable equilibrium. Actually, it should be noted that our data do not allow a precise determination of the border between the dynamical unstable and metastable regions; thus, the number we give here should be looked at as an educated guess inspired by the analogous static classification and based on considerations about the solutions of the gap equation [see eq. (3.36)]

Finally, in section VI we give a brief summary of the results presented in this article and we outline some interesting open questions that need more work before being answered properly.

## II. CUTOFF FIELD THEORY

Let us consider the scalar field operator  $\phi$  and its canonically conjugated momentum  $\pi$  in a  $D$ -dimensional periodic box of size  $L$  and write their Fourier expansion as customary

$$\phi(x) = L^{-D/2} \sum_k \phi_k e^{ik \cdot x}, \quad \phi_k^\dagger = \phi_{-k}$$

$$\pi(x) = L^{-D/2} \sum_k \pi_k e^{ik \cdot x}, \quad \pi_k^\dagger = \pi_{-k}$$

with the wavevectors  $k$  naturally quantized:  $k = (2\pi/L)n$ ,  $n \in \mathbb{Z}^D$ .

The canonical commutation rules are  $[\phi_k, \pi_{-k'}] = i\delta_{kk'}^{(D)}$ , as usual. The introduction of a finite volume should be regarded as a regularization of the infrared properties of the model, which allows to “count” the different field modes and is needed especially in the case of broken symmetry.

To keep control also on the ultraviolet behavior and manage to handle the renormalization procedure properly, we restrict the sums over wavevectors to the points lying within the  $D$ -dimensional sphere of radius  $\Lambda$ , that is  $k^2 \leq \Lambda^2$ , with  $\mathcal{N} = \Lambda L/2\pi$  some large integer. Till both the cut-offs remain finite, we have reduced the original field-theoretical problem to a quantum-mechanical framework with finitely many (of order  $\mathcal{N}^{D-1}$ ) degrees of freedom.

The  $\phi^4$  Hamiltonian is

$$\begin{aligned} H &= \frac{1}{2} \int d^D x \left[ \pi^2 + (\partial\phi)^2 + m_b^2 \phi^2 + \lambda_b \phi^4 \right] = \\ &= \frac{1}{2} \sum_k \left[ \pi_k \pi_{-k} + (k^2 + m_b^2) \phi_k \phi_{-k} \right] + \frac{\lambda}{4} \sum_{k_1, k_2, k_3, k_4} \phi_{k_1} \phi_{k_2} \phi_{k_3} \phi_{k_4} \delta_{k_1+k_2+k_3+k_4,0}^{(D)} \end{aligned}$$

where  $m_b^2$  and  $\lambda_b$  are the bare parameters and depend on the UV cutoff  $\Lambda$  in such a way to guarantee a finite limit  $\Lambda \rightarrow \infty$  for all observable quantities. It should be noted here that, being the theory trivial [14] (as is manifest in the resummed one-loop approximation due to the Landau pole) the ultraviolet cut-off should be kept finite and much smaller than the renormalon singularity. In this case, we must regard the  $\phi^4$  model as an effective low-energy

theory (here low–energy means practically all energies below Planck’s scale, due to the large value of the Landau pole for renormalized coupling constants of order one or less).

We shall work in the wavefunction representation where  $\langle \varphi | \Psi \rangle = \Psi(\varphi)$  and

$$(\phi_0 \Psi)(\varphi) = \varphi_0 \Psi(\varphi), \quad (\pi_0 \Psi)(\varphi) = -i \frac{\partial}{\partial \varphi_0} \Psi(\varphi)$$

while for  $k > 0$  (in lexicographic sense)

$$(\phi_{\pm k} \Psi)(\varphi) = \frac{1}{\sqrt{2}} (\varphi_k \pm i \varphi_{-k}) \Psi(\varphi), \quad (\pi_{\pm k} \Psi)(\varphi) = \frac{1}{\sqrt{2}} \left( -i \frac{\partial}{\partial \varphi_k} \pm \frac{\partial}{\partial \varphi_{-k}} \right) \Psi(\varphi)$$

Notice that by construction the variables  $\varphi_k$  are all real.

In practice, the problem of studying the dynamics of the  $\phi^4$  field out of equilibrium consists now in trying to solve the time-dependent Schrödinger equation given an initial wavefunction  $\Psi(\varphi, t = 0)$  that describes a state of the field far away from the vacuum. This approach could be very well generalized in a straightforward way to mixtures described by density matrices, as done, for instance, in [10,15,16]. Here we shall restrict to pure states, for sake of simplicity and because all relevant aspects of the problem are already present in this case.

We shall consider here the time-dependent Hartree–Fock (tdHF) approach (an improved version with respect to what is presented, for instance, in [13]), being the large  $N$  expansion to leading order treated in another work [8]. In fact these two methods are very closely related (see, for instance in [17]). However, before passing to any approximation, we would like to stress that the following rigorous result can be immediately established in this model with both UV and IR cutoffs.

### A. A rigorous result: the effective potential is convex

This is a well known fact in statistical mechanics, being directly related to stability requirements. It would therefore hold also for the field theory in the Euclidean functional formulation. In our quantum–mechanical context we may proceed as follow. Suppose the field  $\phi$  is coupled to a uniform external source  $J$ . Then the ground state energy  $E_0(J)$  is a concave function of  $J$ , as can be inferred from the negativity of the second order term in  $\Delta J$  of perturbation around any chosen value of  $J$ . Moreover,  $E_0(J)$  is analytic in a finite neighborhood of  $J = 0$ , since  $J\phi$  is a perturbation “small” compared to the quadratic and quartic terms of the Hamiltonian. As a consequence, this effective potential  $V_{\text{eff}}(\bar{\phi}) = E_0(J) - J\bar{\phi}$ ,  $\bar{\phi} = E'_0(J) = \langle \phi \rangle_0$ , that is the Legendre transform of  $E_0(J)$ , is a convex analytic function in a finite neighborhood of  $\bar{\phi} = 0$ . In the infrared limit  $L \rightarrow \infty$ ,  $E_0(J)$  might develop a singularity in  $J = 0$  and  $V_{\text{eff}}(\bar{\phi})$  might flatten around  $\bar{\phi} = 0$ . Of course this possibility would apply in case of spontaneous symmetry breaking, that is for a double–well classical potential. This is a subtle and important point that will play a crucial role later on, even if the effective potential is relevant for the static properties of the model rather than the dynamical evolution out of equilibrium that interests us here. In fact such evolution is governed by the CTP effective action [18,19] and one might expect that, although non–local in time, it asymptotically reduces to a multiple of the effective potential for trajectories of  $\bar{\phi}(t)$  with a fixed point at infinite time. In such case there should exist a one–to–one correspondence between fixed points and minima of the effective potential.

### III. TIME-DEPENDENT HARTREE-FOCK

In order to follow the time evolution of the non-gaussian quantum fluctuations we consider in this section a time-dependent HF approximation capable in principle of describing the dynamics of non-gaussian fluctuations of a single scalar field with  $\phi^4$  interaction.

We examine in this work only states in which the scalar field has a uniform, albeit possibly time-dependent expectation value. In a tdHF approach we may then start from a wavefunction of the factorized form (which would be exact for free fields)

$$\Psi(\varphi) = \psi_0(\varphi_0) \prod_{k>0} \psi_k(\varphi_k, \varphi_{-k}) \quad (3.1)$$

The dependence of  $\psi_k$  on its two arguments cannot be assumed to factorize in general since space translations act as  $SO(2)$  rotations on  $\varphi_k$  and  $\varphi_{-k}$  (hence in case of translation invariance  $\psi_k$  depends only on  $\varphi_k^2 + \varphi_{-k}^2$ ). The approximation consists in assuming this form as valid at all times and imposing the stationarity condition on the action

$$\delta \int dt \langle i\partial_t - H \rangle = 0, \quad \langle \cdot \rangle \equiv \langle \Psi(t) | \cdot | \Psi(t) \rangle \quad (3.2)$$

with respect to variations of the functions  $\psi_k$ . To enforce a uniform expectation value of  $\phi$  we should add a Lagrange multiplier term linear in the single modes expectations  $\langle \varphi_k \rangle$  for  $k \neq 0$ . The multiplier is then fixed at the end to obtain  $\langle \varphi_k \rangle = 0$  for all  $k \neq 0$ . Actually one may verify that this is equivalent to the simpler approach in which  $\langle \varphi_k \rangle$  is set to vanish for all  $k \neq 0$  before any variation. Then the only non trivial expectation value in the Hamiltonian, namely that of the quartic term, assumes the form

$$\begin{aligned} \int d^Dx \langle \phi(x)^4 \rangle &= \frac{1}{L^D} [\langle \varphi_0^4 \rangle - 3\langle \varphi_0^2 \rangle^2] + \frac{3}{2L^D} \sum_{k>0} [\langle (\varphi_k^2 + \varphi_{-k}^2)^2 \rangle - 2(\langle \varphi_k^2 \rangle + \langle \varphi_{-k}^2 \rangle)^2] \\ &\quad + \frac{3}{L^D} \left( \sum_k \langle \varphi_k^2 \rangle \right)^2 \end{aligned} \quad (3.3)$$

Notice that the terms in the first row would cancel completely out for gaussian wavefunctions  $\psi_k$  with zero mean value. The last term, where the sum extends to all wavevectors  $k$ , corresponds instead to the standard mean field replacement  $\langle \phi^4 \rangle \rightarrow 3\langle \phi^2 \rangle^2$ . The total energy of our trial state now reads

$$E = \langle H \rangle = \frac{1}{2} \sum_k \left\langle \frac{\partial^2}{\partial \varphi_k^2} + (k^2 + m_b^2) \varphi_k^2 \right\rangle + \frac{\lambda_b}{4} \int d^Dx \langle \phi(x)^4 \rangle \quad (3.4)$$

and from the variational principle (3.2) we obtain a set of simple Schroedinger equations

$$i\partial_t \psi_k = H_k \psi_k \quad (3.5)$$

$$\begin{aligned} H_0 &= -\frac{1}{2} \frac{\partial^2}{\partial \varphi_0^2} + \frac{1}{2} \omega_0^2 \varphi_0^2 + \frac{\lambda_b}{4L^D} \varphi_0^4 \\ H_k &= -\frac{1}{2} \left( \frac{\partial^2}{\partial \varphi_k^2} + \frac{\partial^2}{\partial \varphi_{-k}^2} \right) + \frac{1}{2} \omega_k^2 (\varphi_k^2 + \varphi_{-k}^2) + \frac{3\lambda_b}{8L^D} (\varphi_k^2 + \varphi_{-k}^2)^2 \end{aligned} \quad (3.6)$$

which are coupled in a mean-field way only through

$$\omega_k^2 = k^2 + m_b^2 + 3\lambda_b \Sigma_k , \quad \Sigma_k = \frac{1}{L^D} \sum_{\substack{q^2 \leq \Lambda^2 \\ q \neq k, -k}} \langle \varphi_q^2 \rangle \quad (3.7)$$

and define the HF time evolution for the theory. By construction this evolution conserves the total energy  $E$  of eq. (3.4).

It should be stressed that in this particular tdHF approximation, beside the mean-field back-reaction term  $\Sigma_k$  of all other modes on  $\omega_k^2$ , we keep also the contribution of the *diagonal* scattering through the diagonal quartic terms. In fact this is why  $\Sigma_k$  has no contribution from the  $k$ -mode itself: in a gaussian approximation for the trial wavefunctions  $\psi_k$  the Hamiltonians  $H_k$  would turn out to be harmonic, the quartic terms being absent in favor of a complete back-reaction

$$\Sigma = \Sigma_k + \frac{\langle \varphi_k^2 \rangle + \langle \varphi_{-k}^2 \rangle}{L^D} = \frac{1}{L^D} \sum_k \langle \varphi_k^2 \rangle \quad (3.8)$$

Of course the quartic self-interaction of the modes as well as the difference between  $\Sigma$  and  $\Sigma_k$  are suppressed by a volume effect and could be neglected in the infrared limit, provided all wavefunctions  $\psi_k$  stays concentrated on mode amplitudes  $\varphi_k$  of order smaller than  $L^{D/2}$ . This is the typical situation when all modes remain microscopic and the volume in the denominators is compensated only through the summation over a number of modes proportional to the volume itself, so that in the limit  $L \rightarrow \infty$  sums are replaced by integrals

$$\Sigma_k \rightarrow \Sigma \rightarrow \int_{k^2 \leq \Lambda^2} \frac{d^D k}{(2\pi)^D} \langle \varphi_k^2 \rangle$$

Indeed we shall apply this picture to all modes with  $k \neq 0$ , while we do expect exceptions for the zero-mode wavefunction  $\psi_0$ .

The treatment of ultraviolet divergences requires particular care, since the HF approximation typically messes things up (see, for instance, [20]). Following the same login of the large  $N$  approximation [1,8,6], we could take as renormalization condition the requirement that the frequencies  $\omega_k^2$  are independent of  $\Lambda$ , assuming that  $m_b^2$  and  $\lambda_b$  are functions of  $\Lambda$  itself and of renormalized  $\Lambda$ -independent parameters  $m^2$  and  $\lambda$  such that

$$\omega_k^2 = k^2 + m^2 + 3\lambda [\Sigma_k]_{\text{finite}} \quad (3.9)$$

where by  $[.]_{\text{finite}}$  we mean the (scheme-dependent) finite part of some possibly ultraviolet divergent quantity. Unfortunately this would not be enough to make the spectrum of energy differences cutoff-independent, because of the bare coupling constant  $\lambda_b$  in front of the quartic terms in  $H_k$  and the difference between  $\Sigma$  and  $\Sigma_k$  [such problem does not exist in large  $N$  because that is a purely gaussian approximation]. Again this would not be a problem whenever these terms become negligible as  $L \rightarrow \infty$ . At any rate, to be ready to handle the cases when this is not actually true and to define an ultraviolet-finite model also at finite volume, we shall by hand modify eq. (3.3) as follows:

$$\begin{aligned} \lambda_b \int d^D x \langle \phi(x)^4 \rangle &= \lambda L^{-D} \left\{ \langle \varphi_0^4 \rangle - 3\langle \varphi_0^2 \rangle^2 + \frac{3}{2} \sum_{k>0} \left[ \langle (\varphi_k^2 + \varphi_{-k}^2)^2 \rangle - 2(\langle \varphi_k^2 \rangle + \langle \varphi_{-k}^2 \rangle)^2 \right] \right\} \\ &\quad + 3\lambda_b L^D \Sigma^2 \end{aligned} \quad (3.10)$$

We keep the bare coupling constant in front of the term containing  $\Sigma^2$  because that part of the hamiltonian is properly renormalized by means of the usual *cactus* resummation [21] which corresponds to the standard HF approximation. On the other hand, within the same approximation, it is not possible to renormalize the part in curly brackets of the equation above, because of the factorized form (3.1) that we have assumed for the wavefunction of the system. In fact, the 4-legs vertices in the curly brackets are diagonal in momentum space; at higher order in the loop expansion, when we contract two or more vertices of this type, no sum over internal loop momenta is produced, so that all higher order perturbation terms are suppressed by volume effects. However, we know that in the complete theory, the wavefunction is not factorized and loops contain all values of momentum. This suggests that, in order to get a finite hamiltonian, we need to introduce in the definition of our model some extra resummation of Feynmann diagrams, that is not automatically contained in this self-consistent HF approach. The only choice consistent with the cactus resummation performed in the two-point function by the HF scheme is the resummation of the 1-loop *fish* diagram in the four-point function. This amounts to the change from  $\lambda_b$  to  $\lambda$  and it is enough to guarantee the ultraviolet finiteness of the hamiltonian through the redefinition

$$H_0 \rightarrow H_0 + \frac{\lambda - \lambda_b}{4L^D} \varphi_0^4, \quad H_k \rightarrow H_k + \frac{3(\lambda - \lambda_b)}{8L^D} (\varphi_k^2 + \varphi_{-k}^2)^2 \quad (3.11)$$

At the same time the frequencies are now related to the widths  $\langle \varphi_{-k}^2 \rangle$  by

$$\begin{aligned} \omega_k^2 &= k^2 + M^2 - 3\lambda L^{-D} (\langle \varphi_k^2 \rangle + \langle \varphi_{-k}^2 \rangle), \quad k > 0 \\ M^2 &\equiv \omega_0^2 + 3\lambda L^{-D} \langle \varphi_0^2 \rangle = m_b^2 + 3\lambda_b \Sigma \end{aligned} \quad (3.12)$$

Apart for  $O(L^{-D})$  corrections,  $M$  plays the role of time-dependent mass for modes with  $k \neq 0$ , in the harmonic approximation.

In this new setup the conserved energy reads

$$E = \sum_{k \geq 0} \langle H_k \rangle - \frac{3}{4} \lambda_b L^D \Sigma^2 + \frac{3}{4} \lambda L^{-D} \left[ \langle \varphi_0^2 \rangle^2 + \sum_{k > 0} (\langle \varphi_k^2 \rangle + \langle \varphi_{-k}^2 \rangle)^2 \right] \quad (3.13)$$

Since the gap-like equations (3.12) are state-dependent, we have to perform the renormalization first for some reference quantum state, that is for some specific collection of wavefunctions  $\psi_k$ ; as soon as  $m_b^2$  and  $\lambda_b$  are determined as functions  $\Lambda$ , ultraviolet finiteness will hold for the entire class of states with the same ultraviolet properties of the reference state. Then an obvious consistency check for our HF approximation is that this class is closed under time evolution.

Rather than a single state, we choose as reference the family of gaussian states parametrized by the uniform expectation value  $\langle \phi(x) \rangle = L^{-D/2} \langle \varphi_0 \rangle = \bar{\phi}$  (recall that we have  $\langle \varphi_k \rangle = 0$  when  $k \neq 0$  by assumption) and such that the HF energy  $E$  is as small as possible for fixed  $\bar{\phi}$ . Then, apart from a translation by  $L^{D/2} \bar{\phi}$  on  $\varphi_0$ , these gaussian  $\psi_k$  are ground state eigenfunctions of the harmonic Hamiltonians obtained from  $H_k$  by dropping the quartic terms. Because of the  $k^2$  in the frequencies we expect these gaussian states to dominate in the ultraviolet limit also at finite volume (as discussed above they should dominate in the infinite-volume limit for any  $k \neq 0$ ). Moreover, since now

$$\langle \varphi_0^2 \rangle = L^D \bar{\phi}^2 + \frac{1}{2\omega_0}, \quad \langle \varphi_{\pm k}^2 \rangle = \frac{1}{2\omega_k}, \quad k \neq 0 \quad (3.14)$$

the relation (3.12) between frequencies and widths turn into the single gap equation

$$M^2 = m_b^2 + 3\lambda_b \left( \bar{\phi}^2 + \frac{1}{2L^D} \sum_{q^2 \leq \Lambda^2} \frac{1}{\sqrt{k^2 + M^2}} \right) \quad (3.15)$$

fixing the self-consistent value of  $M$  as a function of  $\bar{\phi}$ . It should be stressed that (3.12) turns through eq. (3.14) into the gap equation only because of the requirement of energy minimization. Generic  $\psi_k$ , regarded as initial conditions for the Schrödinger equations (3.5), are in principle not subject to any gap equation.

The treatment now follows closely that in the large  $N$  approximation [8], the only difference being in the value of the coupling, now three times larger. In fact, in case of  $O(N)$  symmetry, the quantum fluctuations over a given background  $\langle \phi(x) \rangle = \bar{\phi}$  decompose for each  $k$  into one longitudinal mode, parallel to  $\bar{\phi}$ , and  $N - 1$  transverse modes orthogonal to it; by boson combinatorics the longitudinal mode couples to  $\bar{\phi}$  with strength  $3\lambda_b/N$  and decouple in the  $N \rightarrow \infty$  limit, while the transverse modes couple to  $\bar{\phi}$  with strength  $(N - 1)\lambda_b/N \rightarrow \lambda_b$ ; when  $N = 1$  only the longitudinal mode is there.

As  $L \rightarrow \infty$ ,  $\omega_k^2 \rightarrow k^2 + M^2$  and  $M$  is exactly the physical mass gap. Hence it must be  $\Lambda$ -independent. At finite  $L$  we cannot use this request to determine  $m_b^2$  and  $\lambda_b$ , since, unlike  $M$ , they cannot depend on the size  $L$ . At infinite volume we obtain

$$M^2 = m_b^2 + 3\lambda_b[\bar{\phi}^2 + I_D(M^2, \Lambda)], \quad I_D(z, \Lambda) \equiv \int_{k^2 \leq \Lambda^2} \frac{d^D k}{(2\pi)^D} \frac{1}{2\sqrt{k^2 + z}} \quad (3.16)$$

When  $\bar{\phi} = 0$  this equation fixes the bare mass to be

$$m_b^2 = m^2 - 3\lambda_b I_D(m^2, \Lambda) \quad (3.17)$$

where  $m = M(\bar{\phi} = 0)$  may be identified with the equilibrium physical mass of the scalar particles of the infinite-volume Fock space without symmetry breaking (see below). Now, the coupling constant renormalization follows from the equalities

$$\begin{aligned} M^2 &= m^2 + 3\lambda_b[\bar{\phi}^2 + I_D(M^2, \Lambda) - I_D(m^2, \Lambda)] \\ &= m^2 + 3\lambda \bar{\phi}^2 + 3\lambda \left[ I_D(M^2, \Lambda) - I_D(m^2, \Lambda) \right]_{\text{finite}} \end{aligned} \quad (3.18)$$

and reads when  $D = 3$

$$\frac{\lambda}{\lambda_b} = 1 - \frac{3\lambda}{8\pi^2} \log \frac{2\Lambda}{m\sqrt{e}} \quad (3.19)$$

that is the standard result of the one-loop renormalization group [22]. When  $D = 1$ , that is a  $1 + 1$ -dimensional quantum field theory,  $I_D(M^2, \Lambda) - I_D(m^2, \Lambda)$  is already finite and the dimensionfull coupling constant is not renormalized,  $\lambda_b = \lambda$ .

The Landau pole in  $\lambda_b$  prevents the actual UV limit  $\Lambda \rightarrow \infty$ . Nonetheless, neglecting all inverse powers of the UV cutoff when  $D = 3$ , it is possible to rewrite the gap equation (3.18) as

$$\frac{M^2}{\hat{\lambda}(M)} = \frac{m^2}{\hat{\lambda}(m)} + 3\bar{\phi}^2 \quad (3.20)$$

in terms of the one-loop running coupling constant

$$\hat{\lambda}(\mu) = \lambda \left[ 1 - \frac{3\lambda}{8\pi^2} \log \frac{\mu}{m} \right]^{-1}$$

It is quite clear that the HF states for which the renormalization just defined is sufficient are all those that are gaussian-dominated in the ultraviolet, so that we have [compare to eq. (3.14)]

$$\langle \varphi_{\pm k}^2 \rangle \sim \frac{1}{2\omega_k}, \quad k^2 \sim \Lambda^2, \quad \Lambda \rightarrow \infty \quad (3.21)$$

If this property holds at a certain time, then it should hold at all times, since the Schroedinger equations (3.5) are indeed dominated by the quadratic term for large  $\omega_k$  and  $\omega_k^2 \sim k^2 + \text{const} + O(k^{-1})$  as evident from eq. (3.9). Thus this class of states is indeed closed under time evolution and the parameterizations (3.17) and (3.19) make our tdHF approximation ultraviolet finite. Notice that the requirement (3.21) effectively always imposes a gap equation similar to eq. (3.15) in the deep ultraviolet.

Another simple check of the self-consistency of our approach, including the change in selected places from  $\lambda_b$  to  $\lambda$ , as discussed above, follows from the energy calculation for the gaussian states with  $\langle \phi(x) \rangle = \bar{\phi}$  introduced above. Using eq. (3.4) and the standard replacement of sums by integrals in the infinite volume limit, we find

$$\mathcal{E}(\bar{\phi}) = \lim_{L \rightarrow \infty} \frac{E}{L^D} = \frac{1}{2} \bar{\phi}^2 (M^2 - \lambda \bar{\phi}^2) + \frac{1}{2} \int_{k^2 \leq \Lambda^2} \frac{d^D k}{(2\pi)^D} \sqrt{k^2 + M^2} - \frac{3}{4} \lambda_b [\bar{\phi}^2 + I_D(M^2, \Lambda)]^2$$

where  $M = M(\bar{\phi})$  depends on  $\bar{\phi}$  through the gap equation (3.18). The explicit calculation of the integrals involved shows that the energy density difference  $\mathcal{E}(\bar{\phi}) - \mathcal{E}(0)$  [which for unbroken symmetry is nothing but the effective potential  $V_{\text{eff}}(\bar{\phi})$ ], is indeed finite in the limit  $\Lambda \rightarrow \infty$ , as required by a correct renormalization scheme. Notice that the finiteness of the energy density difference can be shown also by a simpler and more elegant argument, as presented below in section III B. This check would fail instead when  $D = 3$  if only the bare coupling constant  $\lambda_b$  would appear in the last formula.

The tdHF approximation derived above represents a huge simplification with respect to the original problem, but its exact solution still poses itself as a considerable challenge. As a matter of fact, a numerical approach is perfectly possible within the capabilities of modern computers, provided the number of equations (3.5) is kept in the range of few thousands. As will become clear later on, even this numerical workout will turn out not to be really necessary in the form just alluded to, at least for the purposes of this paper.

## A. On symmetry breaking

Quite obviously, in a finite volume and with a UV cutoff there cannot be any symmetry breaking, since the ground state is necessarily unique and symmetric when the number of degrees of freedom is finite [23]. However, we may handily envisage the situation which would imply symmetry breaking when the volume diverges.

Let us first consider the case that we would call of unbroken symmetry. In this case the HF ground state is very close to the member with  $\bar{\phi} = 0$  of the family of gaussian states introduced before. The difference is entirely due to the quartic terms in  $H_k$ . This correction vanish when  $L \rightarrow \infty$ , since all wavefunctions  $\psi_k$  have  $L$ -independent widths, so that one directly obtains the symmetric vacuum state with all the right properties of the vacuum (translation invariance, uniqueness, etc.) upon which a standard scalar massive particle Fock space can be based. The HF approximation then turns out to be equivalent to the resummation of all “cactus diagrams” for the particle self-energy [21]. In a finite volume, the crucial property of this symmetric vacuum is that all frequencies  $\omega_k^2$  are strictly positive. The generalization to non-equilibrium initial states with  $\bar{\phi} \neq 0$  is rather trivial: it amounts to a shift by  $L^{D/2}\bar{\phi}$  on  $\psi_0(\varphi_0)$ . In the limit  $L \rightarrow \infty$  we should express  $\psi_0$  as a function of  $\xi = L^{-D/2}\varphi_0$  so that,  $|\psi_0(\xi)|^2 \rightarrow \delta(\xi - \bar{\phi})$ , while all other wavefunctions  $\psi_k$  will reconstruct the gaussian wavefunctional corresponding to the vacuum  $|0, M\rangle$  of a free massive scalar theory whose mass  $M = M(\phi)$  solves the gap equation (3.18). The absence of  $\psi_0$  in  $|0, M\rangle$  is irrelevant in the infinite volume limit, since  $\langle\varphi_0^2\rangle = L^D\bar{\phi}^2 + \text{terms of order } L^0$ . The effective potential  $V_{\text{eff}}(\bar{\phi}) = \mathcal{E}(\bar{\phi}) - \mathcal{E}(0)$ , where  $\mathcal{E}(\bar{\phi})$  is the lowest energy density at fixed  $\bar{\phi}$  and infinite volume, is manifestly a convex function with a unique minimum in  $\bar{\phi} = 0$ .

Now let us consider a different situation in which one or more of the  $\omega_k^2$  are negative. Quite evidently, this might happen only for  $k$  small enough, due to the  $k^2$  in the gap equation [thus eq. (3.21) remains valid and the ultraviolet renormalization is the same as for unbroken symmetry]. Actually we assume here that only  $\omega_0^2 < 0$ , postponing the general analysis. Now the quartic term in  $H_0$  cannot be neglected as  $L \rightarrow \infty$ , since in the ground state  $\psi_0$  is symmetrically concentrated around the two minima of the potential  $\frac{1}{2}\omega_0^2\varphi_0^2 + \frac{\lambda}{4L^D}\varphi_0^4$ , that is  $\varphi_0 = \pm(-\omega_0^2 L^D / \lambda)^{1/2}$ . If we scale  $\varphi_0$  as  $\varphi_0 = L^{D/2}\xi$  then  $H_0$  becomes

$$H_0 = -\frac{1}{2L^D} \frac{\partial^2}{\partial\xi^2} + \frac{L^D}{2} \left( \omega_0^2 \xi^2 + \frac{\lambda}{2} \xi^4 \right) \quad (3.22)$$

so that the larger  $L$  grows the narrower  $\psi_0(\xi)$  becomes around the two minima  $\xi = \pm(-\omega_0^2/\lambda)^{1/2}$ . In particular  $\langle\xi^2\rangle \rightarrow -\omega_0^2/\lambda$  when  $L \rightarrow \infty$  and  $\langle\varphi_0^2\rangle \simeq L^D\langle\xi^2\rangle$ . Moreover, the energy gap between the ground state of  $H_0$  and its first, odd excited state as well as difference between the relative probability distributions for  $\xi$  vanish exponentially fast in the volume  $L^D$ .

Since by hypothesis all  $\omega_k^2$  with  $k \neq 0$  are strictly positive, the ground state  $\psi_k$  with  $k \neq 0$  are asymptotically gaussian when  $L \rightarrow \infty$  and the relations (3.12) tend to the form

$$\begin{aligned} \omega_k^2 &= k^2 + M^2 \equiv k^2 + m^2 \\ M^2 &= -2\omega_0^2 = m_b^2 + 3\lambda_b(L^{-D}\langle\varphi_0^2\rangle + \Sigma_0) = m_b^2 + 3\lambda_b\omega_0^2 + 3\lambda_b I_D(m^2, \Lambda) \end{aligned}$$

This implies the identification  $\omega_0^2 = -m^2/2$  and the bare mass parameterization

$$m_b^2 = \left(1 - \frac{3}{2}\lambda_b/\lambda\right) m^2 - 3\lambda_b I_D(m^2, \Lambda) \quad (3.23)$$

characteristic of a negative  $\omega_0^2$  [compare to eq. (3.17)], with  $m$  the physical equilibrium mass of the scalar particle, as in the unbroken symmetry case. The coupling constant

renormalization is the same as in eq. (3.19) as may be verified by generalizing to the minimum energy states with given field expectation value  $\bar{\phi}$ ; this minimum energy is nothing but the HF effective potential  $V_{\text{eff}}^{\text{HF}}(\bar{\phi})$ , that the effective potential in this non-gaussian HF approximation; of course, since  $\psi_0$  is no longer asymptotically gaussian, we cannot simply shift it by  $L^{D/2}\bar{\phi}$  but, due to the concentration of  $\psi_0$  on classical minima as  $L \rightarrow \infty$ , one readily finds that  $V_{\text{eff}}(\bar{\phi})$  is the convex envelope of the classical potential, that is its Maxwell construction. Hence we find

$$\langle \varphi_0^2 \rangle \underset{L \rightarrow \infty}{\sim} \begin{cases} -L^D \omega_0^2 / \lambda , & \lambda \bar{\phi}^2 \leq -\omega_0^2 \\ L^D \bar{\phi}^2 , & \lambda \bar{\phi}^2 > -\omega_0^2 \end{cases}$$

and the gap equation for the  $\bar{\phi}$ -dependent mass  $M$  can be written, in terms of the step function  $\Theta$  and the extremal ground state field expectation value  $v = m/\sqrt{2\lambda}$ ,

$$M^2 = m^2 + 3\lambda_b(\bar{\phi}^2 - v^2)\Theta(\bar{\phi}^2 - v^2) + 3\lambda_b [I_D(M^2, \Lambda) - I_D(m^2, \Lambda)] \quad (3.24)$$

We see that the specific bare mass parameterization (3.23) guarantees the non-renormalization of the tree-level relation  $v^2 = m^2/2\lambda$  ensuing from the typical symmetry breaking classical potential  $V(\phi) = \frac{1}{4}\lambda(\phi^2 - v^2)^2$ . With the same finite part prescription as in eq. (3.18), the gap equation (3.24) leads to the standard coupling constant renormalization (3.19) when  $D = 3$ .

In terms of the probability distributions  $|\psi_0(\xi)|^2$  for the scaled amplitude  $\xi = L^{-D/2}\varphi_0$ , the Maxwell construction corresponds to the limiting form

$$|\psi_0(\xi)|^2 \underset{L \rightarrow \infty}{\sim} \begin{cases} \frac{1}{2}(1 + \bar{\phi}/v)\delta(\xi - v) + \frac{1}{2}(1 - \bar{\phi}/v)\delta(\xi + v) , & \bar{\phi}^2 \leq v^2 \\ \delta(\xi - \bar{\phi}) , & \bar{\phi}^2 > v^2 \end{cases} \quad (3.25)$$

On the other hand, if  $\omega_0^2$  is indeed the only negative squared frequency, the  $k \neq 0$  part of this minimum energy state with arbitrary  $\bar{\phi} = \langle \phi(x) \rangle$  is better and better approximated as  $\Lambda \rightarrow \infty$  by the same gaussian state  $|0, M\rangle$  of the unbroken symmetry state. Only the effective mass  $M$  has a different dependence  $M(\bar{\phi})$ , as given by the gap equation (3.24) proper of broken symmetry.

At infinite volume we may write

$$\langle \varphi_k^2 \rangle = C(\bar{\phi}) \delta^{(D)}(k) + \frac{1}{2\sqrt{k^2 + M^2}}$$

where  $C(\bar{\phi}) = \bar{\phi}^2$  in case of unbroken symmetry (that is  $\omega_0^2 > 0$ ), while  $C(\bar{\phi}) = \max(v^2, \bar{\phi}^2)$  when  $\omega_0^2 < 0$ . This corresponds to the field correlation in space

$$\langle \phi(x)\phi(y) \rangle = \int \frac{d^D k}{(2\pi)^D} \langle \varphi_k^2 \rangle e^{ik \cdot (x-y)} = C(\bar{\phi}) + \Delta_D(x-y, M)$$

where  $\Delta_D(x-y, M)$  is the massive free field equal-time two points function in  $D$  space dimensions, with self-consistent mass  $M$ . The requirement of clustering

$$\langle \phi(x)\phi(y) \rangle \rightarrow \langle \phi(x) \rangle^2 = v^2$$

contradicts the infinite volume limit of

$$\langle \phi(x) \rangle = L^{-D/2} \sum_k \langle \phi_k \rangle e^{ik \cdot x} = \langle \varphi_0 \rangle = \bar{\phi}$$

except at the two extremal points  $\bar{\phi} = \pm v$ . In fact we know that the  $L \rightarrow \infty$  limit of the finite volume states with  $\bar{\phi}^2 < v^2$  violate clustering, because the two peaks of  $\psi_0(\xi)$  have vanishing overlap in the limit and the first excited state becomes degenerate with the vacuum: this implies that the relative Hilbert space splits into two orthogonal Fock sectors each exhibiting symmetry breaking,  $\langle \phi(x) \rangle = \pm v$ , and corresponding to the two independent equal weight linear combinations of the two degenerate vacuum states. The true vacuum is either one of these symmetry broken states. Since the two Fock sectors are not only orthogonal, but also superselected (no local observable interpolates between them), linear combinations of any pair of vectors from the two sectors are not distinguishable from mixtures of states and clustering cannot hold in non-pure phases. It is perhaps worth noticing also that the Maxwell construction for the effective potential, in the infinite volume limit, is just a straightforward manifestation of this fact and holds true, as such, beyond the HF approximation.

To further clarify this point and in view of subsequent applications, let us consider the probability distribution for the smeared field  $\phi_f = \int d^D x \phi(x) f(x)$ , where

$$f(x) = f(-x) = \frac{1}{L^D} \sum_k f_k e^{ik \cdot x} \underset{L \rightarrow \infty}{\sim} \int \frac{d^D k}{(2\pi)^D} \tilde{f}(k) e^{ik \cdot x}$$

is a smooth real function with  $\int d^D x f(x) = 1$  (*i.e.*  $f_0 = 1$ ) localized around the origin (which is good as any other point owing to translation invariance). Neglecting in the infinite volume limit the quartic corrections for all modes with  $k \neq 0$ , so that the corresponding ground state wavefunctions are asymptotically gaussian, this probability distribution evaluates to

$$\Pr(u < \phi_f < u + du) = \frac{du}{(2\pi\Sigma_f)^{1/2}} \int_{-\infty}^{+\infty} d\xi |\psi_0(\xi)|^2 \exp\left\{-\frac{(u - \xi)^2}{2\Sigma_f}\right\}$$

where

$$\Sigma_f = \sum_{k \neq 0} \langle \varphi_k^2 \rangle f_k^2 \underset{L \rightarrow \infty}{\sim} \int \frac{d^D k}{(2\pi)^D} \frac{\tilde{f}(k)^2}{2\sqrt{k^2 + m^2}}$$

In the unbroken symmetry case we have  $|\psi_0(\xi)|^2 \sim \delta(\xi - \bar{\phi})$  as  $L \rightarrow \infty$ , while the limiting form (3.25) holds for broken symmetry. Thus we obtain

$$\Pr(u < \phi_f < u + du) = p_f(u - \bar{\phi}) du, \quad p_f(u) \equiv (2\pi\Sigma_f)^{-1/2} \exp\left(\frac{-u^2}{2\Sigma_f}\right)$$

for unbroken symmetry and

$$\Pr(u < \phi_f < u + du) = \begin{cases} \frac{1}{2}(1 + \bar{\phi}/v) p_f(u - v) du + \frac{1}{2}(1 - \bar{\phi}/v) p_f(u + v) du, & \bar{\phi}^2 \leq v^2 \\ p_f(u - \bar{\phi}) du, & \bar{\phi}^2 > v^2 \end{cases}$$

for broken symmetry. Notice that the momentum integration in the expression for  $\Sigma_f$  needs no longer an ultraviolet cutoff; of course in the limit of delta-like test function  $f(x)$ ,  $\Sigma_f$

diverges and  $p_f(u)$  flattens down to zero. The important observation is that  $\Pr(u < \phi_f < u + du)$  has always a single peak centered in  $u = \bar{\phi}$  for unbroken symmetry, while for broken symmetry it shows two peaks for  $\bar{\phi}^2 \leq v^2$  and  $\Sigma_f$  small enough. For instance, if  $\bar{\phi} = 0$ , then there are two peaks for  $\Sigma_f < v^2$  [implying that  $\tilde{f}(k)$  has a significant support only up to wavevector  $k$  of order  $v$ , when  $D = 3$ , or  $m \exp(\text{const } v^2)$  when  $D = 1$ ].

To end the discussion on symmetry breaking, we may now verify the validity of the assumption that only  $\omega_0^2$  is negative. In fact, to any squared frequency  $\omega_k^2$  (with  $k \neq 0$ ) that stays strictly negative as  $L \rightarrow \infty$  there corresponds a wavefunction  $\psi_k$  that concentrates on  $\varphi_k^2 + \varphi_{-k}^2 = -\omega_k^2 L^D / \lambda$ ; then eqs. (3.12) implies  $-2\omega_k^2 = k^2 + m^2$  for such frequencies, while  $\omega_k^2 = k^2 + m^2$  for all frequencies with positive squares; if there is a macroscopic number of negative  $\omega_k^2$  (that is a number of order  $L^D$ ), then the expression for  $\omega_0^2$  in eq. (3.12) will contain a positive term of order  $L^D$  in the r.h.s., clearly incompatible with the requirements that  $\omega_0^2 < 0$  and  $m_b^2$  be independent of  $L$ ; if the number of negative  $\omega_k^2$  is not macroscopic, then the largest wavevector with a negative squared frequency tends to zero as  $L \rightarrow \infty$  (the negative  $\omega_k^2$  clearly pile in the infrared) and the situation is equivalent, if not identical, to that discussed above with only  $\omega_0^2 < 0$ .

## B. Out-of-equilibrium dynamics

We considered above the lowest energy states with a predefinite uniform field expectation value,  $\langle \phi(x) \rangle = \bar{\phi}$ , and established how they drastically simplify in the infinite volume limit. For generic  $\bar{\phi}$  these states are not stationary and will evolve in time. By hypothesis  $\psi_k$  is the ground state eigenfunction of  $H_k$  when  $k > 0$ , and therefore  $|\psi_k|^2$  would be stationary for constant  $\omega_k$ , but  $\psi_0$  is not an eigenfunction of  $H_0$  unless  $\bar{\phi} = 0$ . As soon as  $|\psi_k|^2$  starts changing,  $\langle \varphi_0^2 \rangle$  changes and so do all frequencies  $\omega_k$  which are coupled to it by the eqs. (3.12). Thus the change propagates to all wavefunctions. The difficult task of studying this dynamics can be simplified with the following scheme, that we might call *gaussian approximation*. We first describe it and discuss its validity later on.

Let us assume the usual gaussian form for the initial state [see eq. (3.14) and the discussion following it]. We know that it is a good approximation to the lowest energy state with given  $\langle \varphi_0 \rangle$  for unbroken symmetry, while it fails to be so for broken symmetry, only as far as  $\psi_0$  is concerned, unless  $\bar{\phi}^2 \geq v^2$ . At any rate this is an acceptable initial state: the question is about its time evolution. Suppose we adopt the harmonic approximation for all  $H_k$  with  $k > 0$  by dropping the quartic term. This approximation will turn out to be valid only if the width of  $\psi_k$  do not grow up to the order  $L^D$  (by symmetry the center will stay in the origin). In practice we are now dealing with a collection of harmonic oscillators with time-dependent frequencies and the treatment is quite elementary: consider the simplest example of one quantum degree of freedom described by the gaussian wavefunction

$$\psi(q, t) = \frac{1}{(2\pi\sigma^2)^{1/4}} \exp \left[ -\frac{1}{2} \left( \frac{1}{2\sigma^2} - i\frac{s}{\sigma} \right) q^2 \right]$$

where  $s$  and  $\sigma$  are time-dependent. If the dynamics is determined by the time-dependent harmonic hamiltonian  $\frac{1}{2}[-\partial_q^2 + \omega(t)^2 q^2]$ , then the Schrödinger equation is solved exactly

provided that  $s$  and  $\sigma$  satisfy the classical Hamilton equations

$$\dot{\sigma} = s, \quad \dot{s} = -\omega^2 \sigma + \frac{1}{4\sigma^3}$$

It is not difficult to trace the “centrifugal” force  $(4\sigma)^{-3}$  which prevents the vanishing of  $\sigma$  to Heisenberg uncertainty principle [2,15].

The extension to our case with many degrees of freedom is straightforward and we find the following system of equations

$$i\frac{\partial}{\partial t}\psi_0 = H_0\psi_0, \quad \frac{d^2\sigma_k}{dt^2} = -\omega_k^2 \sigma_k + \frac{1}{4\sigma_k^3}, \quad k > 0 \quad (3.26)$$

coupled in a mean-field way by the relations (3.12), which now read

$$\begin{aligned} \omega_k^2 &= k^2 + M^2 - 6\lambda L^{-D} \sigma_k^2, \quad k > 0 \\ M^2 &= m_b^2 + 3\lambda_b \left( L^{-D} \langle \varphi_0^2 \rangle + \Sigma_0 \right), \quad \Sigma_0 = \frac{1}{L^D} \sum_{k \neq 0} \sigma_k^2 \end{aligned} \quad (3.27)$$

This stage of a truly quantum zero-mode and classical modes with  $k > 0$  does not appear fully consistent, since for large volumes some type of classical or gaussian approximation should be considered for  $\varphi_0$  too. We may proceed in two (soon to be proven equivalent) ways:

1. We shift  $\varphi_0 = L^{D/2} \bar{\phi} + \eta_0$  and then deal with the quantum mode  $\eta_0$  in the gaussian approximation, taking into account that we must have  $\langle \eta_0 \rangle = 0$  at all times. This is most easily accomplished in the Heisenberg picture rather than in the Schroedinger one adopted above. In any case we find that the quantum dynamics of  $\varphi_0$  is equivalent to the classical dynamics of  $\bar{\phi}$  and  $\sigma_0 \equiv \langle \eta_0^2 \rangle^{1/2}$  described by the ordinary differential equations

$$\frac{d^2\bar{\phi}}{dt^2} = -\omega_0^2 \bar{\phi} - \lambda \bar{\phi}^3, \quad \frac{d^2\sigma_0}{dt^2} = -\omega_0^2 \sigma_0 + \frac{1}{4\sigma_0^3} \quad (3.28)$$

where  $\omega_0^2 = M^2 - 3\lambda L^{-D} \langle \varphi_0^2 \rangle$  and  $\langle \varphi_0^2 \rangle = L^D \bar{\phi}^2 + \sigma_0^2$ .

2. We rescale  $\varphi_0 = L^{D/2} \xi$  right away, so that  $H_0$  takes the form of eq. (3.22). Then  $L \rightarrow \infty$  is the classical limit such that  $\psi_0(\xi)$  concentrates on  $\xi = \bar{\phi}$  which evolves according to the first of the classical equations in (3.28). Since now there is no width associated to the zero-mode,  $\bar{\phi}$  is coupled only to the widths  $\sigma_k$  with  $k \neq 0$  by  $\omega_0^2 = M^2 - 3\lambda \bar{\phi}^2$ , while  $M^2 = m_b^2 + 3\lambda_b (\bar{\phi}^2 + \Sigma_0)$ .

It is quite evident that these two approaches are completely equivalent in the infinite volume limit, and both are good approximation to the original tdHF Schrödinger equations, at least provided that  $\sigma_0^2$  stays such that  $L^{-D} \sigma_0^2$  vanishes in the limit for any time. In this case we have the evolution equations

$$\frac{d^2\bar{\phi}}{dt^2} = (2\lambda \bar{\phi}^2 - M^2) \bar{\phi}, \quad \frac{d^2\sigma_k}{dt^2} = -(k^2 + M^2) \sigma_k + \frac{1}{4\sigma_k^3} \quad (3.29)$$

mean-field coupled by the  $L \rightarrow \infty$  limit of eqs. (3.27), namely

$$M^2 = m^2 + 3\lambda_b [\bar{\phi}^2 + \Sigma - I_D(m^2, \Lambda)] \quad (3.30)$$

for unbroken symmetry [that is  $m_b^2$  as in eq. (3.17)] or

$$M^2 = m^2 + 3\lambda_b [\bar{\phi}^2 - v^2 + \Sigma - I_D(m^2, \Lambda)] , \quad m^2 = 2\lambda v^2 \quad (3.31)$$

for broken symmetry [that is  $m_b^2$  as in eq. (3.23)]. In any case we define

$$\Sigma = \frac{1}{L^D} \sum_k \sigma_k^2 \underset{L \rightarrow \infty}{\sim} \int_{k^2 \leq \Lambda^2} \frac{d^D k}{(2\pi)^D} \sigma_k^2$$

as the sum, or integral, over all microscopic gaussian widths [N.B.:this definition differs from that given before in eq. (3.8) by the classical term  $\bar{\phi}^2$ ]. Remarkably, the equations of motion (3.29) are completely independent of the ultraviolet cut-off and this is a direct consequence of the substitution (3.11). Had we kept the bare coupling constant everywhere in the expression (3.10), we would now have  $\lambda_b$  also in front of the  $\bar{\phi}^3$  in the r.h.s. of the first of the two equations (3.29) [cfr., for instance, ref. [1]].

The conserved HF energy (density) corresponding to these equations of motion reads

$$\begin{aligned} \mathcal{E} &= \mathcal{T} + \mathcal{V} , \quad \mathcal{T} = \frac{1}{2} (\dot{\bar{\phi}})^2 + \frac{1}{2L^D} \sum_k \dot{\sigma}_k^2 \\ \mathcal{V} &= \frac{1}{2L^D} \sum_k \left( k^2 \sigma_k^2 + \frac{1}{4\sigma_k^2} \right) + \frac{1}{2} m_b^2 (\bar{\phi}^2 + \Sigma) + \frac{3}{4} \lambda_b (\bar{\phi}^2 + \Sigma)^2 - \frac{1}{2} \lambda \bar{\phi}^4 \end{aligned} \quad (3.32)$$

Up to additive constants and terms vanishing in the infinite volume limit, this expression agrees with the general HF energy of eq. (3.13) for gaussian wavefunctions. It holds both for unbroken and broken symmetry, the only difference being in the parameterization of the bare mass in terms of UV cutoff and physical mass, eqs. (3.17) and (3.23). The similarity to the energy functional of the large  $N$  approach is evident; the only difference, apart from the obvious fact that  $\bar{\phi}$  is a single scalar rather than a  $O(n)$  vector, is in the mean-field coupling  $\sigma_k \bar{\phi}$  and  $\sigma_k \Sigma$ , due to different coupling strength of transverse and longitudinal modes (cfr. ref. [8]).

This difference between the HF approach for discrete symmetry (*i.e*  $N = 1$ ) and the large  $N$  method for the continuous  $O(N)$ -symmetry is not very relevant if the symmetry is unbroken [it does imply however a significantly slower dissipation to the modes of the background energy density]. On the other hand it has a drastic consequence on the equilibrium properties and on the out-of-equilibrium dynamics in case of broken symmetry (see below), since massless Goldstone bosons appear in the large  $N$  approach, while the HF treatment of the discrete symmetry case must exhibits a mass also in the broken symmetry phase.

The analysis of physically viable initial conditions proceeds exactly as in the large  $N$  approach [8] and will not be repeated here, except for an important observation in case of broken symmetry. The formal energy minimization w.r.t.  $\sigma_k$  at fixed  $\bar{\phi}$  leads again to eqs.

$$\dot{\sigma}_k = 0 , \quad \sigma_k^2 = \frac{1}{2\sqrt{k^2 + M^2}} \quad (3.33)$$

and again these are acceptable initial conditions only if the gap equation that follows from eq. (3.31) in the  $L \rightarrow \infty$  limit, namely

$$M^2 = m^2 + 3\lambda_b [\bar{\phi}^2 - v^2 + I_D(M^2, \Lambda) - I_D(m^2, \Lambda)] \quad (3.34)$$

admits a nonnegative, physical solution for  $M^2$ . Notice that there is no step function in eq. (3.34), unlike the static case of eq. (3.24), because  $\sigma_0^2$  was assumed to be microscopic, so that the infinite volume  $\sigma_k^2$  has no delta-like singularity in  $k = 0$ . Hence  $M = m$  solves eq. (3.34) only at the extremal points  $\bar{\phi} = \pm v$ , while it was the solution of the static gap equation (3.24) throughout the Maxwell region  $-v \leq \bar{\phi} \leq v$ . The important observation is that eq. (3.34) admits a positive solution for  $M^2$  also within the Maxwell region. In fact it can be written, neglecting as usual the inverse-power corrections in the UV cutoff

$$\frac{M^2}{\hat{\lambda}(M)} = \frac{m^2}{\lambda} + 3(\bar{\phi}^2 - v^2) = 3\bar{\phi}^2 - v^2 \quad (3.35)$$

and there exists indeed a positive solution  $M^2$  smoothly connected to the ground state,  $\bar{\phi}^2 = v^2$  and  $M^2 = m^2$ , whenever  $\bar{\phi}^2 \geq v^2/3$ . The two intervals  $v^2 \geq \bar{\phi}^2 \geq v^2/3$  correspond indeed to the metastability regions, while  $\bar{\phi}^2 < v^2/3$  is the spinodal region, associated to a classical potential proportional to  $(\bar{\phi}^2 - v^2)^2$ . This is another effect of the different coupling of transverse and longitudinal modes: in the large  $N$  approach there are no metastability regions and the spinodal region coincides with the Maxwell one. As in the large  $N$  approach in the spinodal interval there is no energy minimization possible, at fixed background and for microscopic widths, so that a modified form of the gap equation

$$M^2 = m^2 + 3\lambda_b \left[ \bar{\phi}^2 - v^2 + \frac{1}{L^D} \sum_{k^2 < |M^2|} \sigma_k^2 + \frac{1}{L^D} \sum_{k^2 > |M^2|} \frac{1}{2\sqrt{k^2 - |M^2|}} - I_D(0, \Lambda) \right] \quad (3.36)$$

should be applied to determine ultraviolet-finite initial conditions.

The main question now is: how will the gaussian widths  $\sigma_k$  grow with time, and in particular how will  $\sigma_0$  grow in case of method 1 above, when we start from initial conditions where all widths are microscopic? For the gaussian approximation to remain valid through time, all  $\sigma_k$ , and in particular  $\sigma_0$ , must at least not become macroscopic. In fact we have already positively answered this question in the large  $N$  approach [8] and the HF equations (3.29) do not differ so much to expect the contrary now. In particular, if we consider the special initial condition  $\dot{\bar{\phi}} = \dot{\bar{\phi}} = 0$ , the dynamics of the widths is identical to that in the large  $N$  approach, apart from the rescaling by a factor of three of the coupling constant.

In fact, if we look at the time evolution of the zero-mode amplitude  $\sigma_0$  [see Fig. 1], we can see the presence of the time-scale  $\tau_L$  at which finite volume effects start to manifest. The time scale  $\tau_L$  turns out to be proportional to the linear size of the box  $L$  and its presence prevents  $\sigma_0$  from growing to macroscopic values. Thus our HF approximation confirms the large  $N$  approach in the following sense: even if one considers in the variational ansatz the possibility of non-gaussian wavefunctionals, the time evolution from gaussian and microscopic initial conditions is effectively restricted for large volumes to non-macroscopic gaussians.

Strictly speaking, however, this might well not be enough, since the infrared fluctuations do grow beyond the microscopic size to become of order  $L$  [see Fig. 2, where the evolution

of the mode with momentum  $k = 2\pi/L$  is plotted]. Then the quartic term in the low- $k$  Hamiltonians  $H_k$  is of order  $L$  and therefore it is not negligible by itself in the  $L \rightarrow \infty$  limit, but only when compared to the quadratic term, which *for a fixed  $\omega_k^2$  of order 1* would be of order  $L^2$ . But we know that, when  $\bar{\phi} = 0$ , after the spinodal time and before the  $\tau_L$ , the effective squared mass  $M^2$  oscillates around zero with amplitude decreasing as  $t^{-1}$  and a frequency fixed by the largest spinodal wavevector. In practice it is “zero on average” and this reflects itself in the average linear growth of the zero-mode fluctuations and, more generally, in the average harmonic motion of the other widths with non-zero wavevectors. In particular the modes with small wavevectors of order  $L^{-1}$  feel an average harmonic potential with  $\omega_k^2$  of order  $L^{-2}$ . This completely compensates the amplitude of the mode itself, so that the quadratic term in the low- $k$  Hamiltonians  $H_k$  is of order  $L^0$ , much smaller than the quartic term that was neglected beforehand in the gaussians approximation. Clearly the approximation itself no longer appears fully justified and a more delicate analysis is required. We intend to return on this issue in a future work, restricting ourselves in the next section to the gaussians approximation.

#### IV. LATE-TIME EVOLUTION AND DYNAMICAL MAXWELL CONSTRUCTION

By definition, the gaussian approximation of the effective potential  $V_{\text{eff}}(\bar{\phi})$  coincides with the infinite-volume limit of the potential energy  $\mathcal{V}(\bar{\phi}, \{\sigma_k\})$  of eq. (3.32) when the widths are of the  $\bar{\phi}$ -dependent, energy-minimizing form (3.33) with the gap equation for  $M^2$  admitting a nonnegative solution. As we have seen, this holds true in the unbroken symmetry case for any value of the background  $\bar{\phi}$ , so that the gaussian  $V_{\text{eff}}$  is identical to the HF one, since all wavefunctions  $\psi_k$  are asymptotically gaussians as  $L \rightarrow \infty$ . In the presence of symmetry breaking instead, this agreement holds true only for  $\bar{\phi}^2 \geq v^2$ ; for  $v^2/3 \leq \bar{\phi}^2 < v^2$  the gaussian  $V_{\text{eff}}$  exists but is larger than the HF potential  $V_{\text{eff}}^{\text{HF}}$ , which is already flat. In fact, for any  $\bar{\phi}^2 \geq v^2/3$ , we may write the gaussian  $V_{\text{eff}}$  as

$$V_{\text{eff}}(\bar{\phi}) = V_{\text{eff}}(-\bar{\phi}) = V_{\text{eff}}(v) + \int_v^{|\bar{\phi}|} du u [M(u)^2 - 2\lambda u^2]$$

where  $M(u)^2$  solves the gap equation (3.35), namely  $M(u)^2 = \hat{\lambda}(M(u))(3u^2 - v^2)$ . In each of the two disjoint regions of definition this potential is smooth and convex, with unique minima in  $+v$  and  $-v$ , respectively. These appear therefore as regions of metastability (states which are only locally stable in the presence of a suitable uniform external source). The HF effective potential is identical for  $\bar{\phi}^2 \geq v^2$ , while it takes the constant value  $V_{\text{eff}}(v)$  throughout the internal region  $\bar{\phi}^2 < v^2$ . It is based on truly stable (not only metastable) states. The gaussian  $V_{\text{eff}}$  cannot be defined in the spinodal region  $\bar{\phi}^2 < v^2/3$ , where the gap equation does not admit a nonnegative solution in the physical region far away from the Landau pole.

Let us first compare this HF situation with that of large  $N$  [8]. There the different coupling of the transverse modes, three times smaller than the HF longitudinal coupling, has two main consequences at the static level: the gap equation similar to (3.35) does not admit nonnegative solutions for  $\bar{\phi}^2 < v^2$ , so that the spinodal region coincides with the region in

which the effective potential is flat, and the physical mass vanishes. The out-of-equilibrium counterpart of this is the dynamical Maxwell construction: when the initial conditions are such that  $\bar{\phi}^2$  has a limit for  $t \rightarrow \infty$ , the set of all possible asymptotic values exactly covers the flatness region (and the effective mass vanishes in the limit). In practice this means that  $|\bar{\phi}|$  is not the true dynamical order parameter, whose large time limit coincides with  $v$ , the equilibrium field expectation value in a pure phase. Rather, one should consider as order parameter the renormalized local (squared) width

$$\lim_{N \rightarrow \infty} \frac{\langle \phi(x) \cdot \phi(x) \rangle_R}{N} = \bar{\phi}^2 + \Sigma_R = v^2 + \frac{M^2}{\lambda}$$

where the last equality follows from the definition itself of the effective mass  $M$  (see ref. [8]). Since  $M$  vanishes as  $t \rightarrow \infty$  when  $\bar{\phi}^2$  tends to a limit within the flatness region, we find the renormalized local width tends to the correct value  $v$  which characterizes the broken symmetry phase, that is the bottom of the classical potential. We may say that the spinodal region, perturbatively unstable, at the non-perturbative level corresponds to metastable states, all reachable through the asymptotic time evolution with a vanishing effective mass.

In the HF approximation, where at the static level the spinodal region  $\bar{\phi}^2 < v^2/3$  is smaller than the flatness region  $\bar{\phi}^2 < v^2$ , the situation is rather different. Our numerical solution shows that,  $\bar{\phi}$  oscillates around a certain value  $\bar{\phi}_\infty$  with an amplitude that decreases very slowly. As in large  $N$ , the asymptotic value  $\bar{\phi}_\infty$  depends on the initial value  $\bar{\phi}(0)$ . But, if the background  $\bar{\phi}$  starts with zero velocity from a non-zero value inside the spinodal interval, then it always leaves this region and eventually oscillates around a point between the spinodal point  $v/\sqrt{3}$  and the minimum of the tree level potential  $v$  (see Fig.s 3 and 4). In other words, if we start with a  $\bar{\phi}$  in the interval  $[-v, v]$ , except the origin, we end up with a  $\bar{\phi}_\infty$  in the restricted interval  $[-v, -v/\sqrt{3}] \cup [v/\sqrt{3}, v]$ . The spinodal region is completely forbidden for the late time evolution of the mean field, as is expected for an unstable region. We stress that we are dealing with true fixed points of the asymptotic evolution since the force term on the mean field [cfr. eq. (3.29),  $f = (2\lambda \bar{\phi}^2 - M^2) \bar{\phi}$ ] does vanish in the limit. In fact its time average  $\bar{f} = \int^T f(t) dt / T$  tends to zero as  $T$  grows and its mean squared fluctuations around  $\bar{f}$  decreases towards zero, although very slowly (see Fig.s 5 and 6). Moreover, for  $N = 1$  the order parameter reads as  $t \rightarrow \infty$

$$\langle \phi(x)^2 \rangle_R = \bar{\phi}^2 + \Sigma_R = \frac{v^2}{3} + \frac{M^2}{3\lambda}, \quad \Sigma_R = \frac{v^2 - \bar{\phi}^2}{3} \quad (4.1)$$

where the last equality is valid for the asymptotic values and follows from the vanishing of the force term  $f$ . From the last formula we see that when  $\bar{\phi} = 0$  at the beginning, and then at all times, the renormalized back-reaction tends to  $v^2/3$ , not  $v^2$ . It “stops at the spinodal line”. The same picture applies for a long time, all during the “slow rolling down” (see section V), to evolutions that start close enough to  $\bar{\phi} = 0$ . This fact is at the basis of the so-called *spinodal inflation* [24].

In any case, the dynamical Maxwell construction, either complete or partial, poses an interesting question by itself. In fact it is not at all trivial that the effective potential, in any of the approximation previously discussed, does bear relevance on the asymptotic behavior of the infinite-volume system whenever a fixed point is approached. Strictly speaking in

fact, even in such a special case it is not directly related to the dynamics, since it is obtained from a static minimization of the total energy at fixed mean field, while the energy is not at its minimum at the initial time and is exactly conserved in the evolution. On the other hand, if a solution of the equations of motion (3.29) exists in which the background  $\bar{\phi}$  tends to a constant  $\bar{\phi}_\infty$  as  $t \rightarrow \infty$ , one might expect that the effective action (which however is nonlocal in time) somehow reduces to a (infinite) multiple of the effective potential, so that  $\bar{\phi}_\infty$  should be an extremal of the effective potential. This is still an open question that deserves further analytic studies and numerical confirmation.

It is worth noticing also that when the field starts very close to the top of the potential hill, it remains there for a very long time and evolves through a very slow rolling down, before beginning a damped oscillatory motion around a point in the metastability region. During the slow roll period,  $M^2$  oscillates around zero with decreasing amplitude and the “phenomenology” is very similar to the evolution from symmetric initial conditions, as can be seen comparing Fig.s 7 and 8. Fig. 9 shows the evolution of the zero mode amplitude in case of a very slow rolling down. In such a case, after a very short (compared to the time scale of the figure) period of exponential growth (the spinodal time), the quantum fluctuations start an almost linear growth, very similar to the evolution starting from a completely symmetric initial state. This, obviously, corresponds to the vanishing of the effective mass. In the meanwhile,  $\bar{\phi}$  keeps growing and rolling down the potential hill with increasing speed towards the minimum of the classical potential, eventually entering the metastable region. At that time, the effective mass starts to increase again and the zero mode stops its linear growth, turns down and enters a phase of “wild” evolution. This time scale, let us call it  $\tau_{srd}$ , depends on the initial value of the condensate: the smaller  $\bar{\phi}(t=0)$  is, the longer  $\tau_{srd}$  will be. We find numerically that  $\tau_{srd} \propto (\bar{\phi}(t=0))^{1/2}$ .

If we now study the dynamics in finite volume, starting from condensates different from zero, we will find a competition between  $\tau_{srd}$  and  $\tau_L$ , the time scale characteristic of the finite volume effects, that is proportional to the linear size of the box we put the system in. Fig. 10 shows clearly that when  $L/2\pi = 100$  and  $\bar{\phi} = 10^{-5}$ , we have  $\tau_{srd} \sim \tau_L$ . In any case, either one or the other effect will prevent the zero mode amplitude from growing to macroscopic values for any initial condition we may start with.

It should be noted, also, that the presence of the time scale  $\tau_{srd}$  does not solve the internal inconsistency of the gaussian approximation described above in section III B. In fact, for any fixed value  $L$  for the linear size of the system, we can find a whole interval of initial conditions for the mean field, which leave enough time to the fluctuations for growing to order  $L$ , much before the field itself had rolled down towards one of the minima of the classical potential. For those particular evolutions, we would need to consider the quartic terms in the hamiltonians that the gaussian approximation neglects, as already explained.

In addition, there will be also initial conditions for which  $\tau_L > \tau_{srd}$ . In that case, the effective mass soon starts oscillating around positive values and it is reasonable to think that it will take a much longer time than  $\tau_L$  for the finite volume effects to manifest. In [8] we have interpreted the proportionality between  $\tau_L$  and  $L$  as an auto interference effect (due to periodic boundary conditions) suffered by a Goldstone boson wave, traveling at speed of light, at the moment it reaches the borders of the cubic box. Here, the massless wave we have in the early phase of the evolution, rapidly acquires a positive mass, as soon as the condensate rolls down; this decelerates the wave’s propagation and delays the onset of finite

volume effects. The gaussian approximation appears to be fully consistent when we limit ourselves to the evolution of these particular configurations.

## V. NUMERICAL ANALYSIS

We discuss in this section the asymptotic behavior of the dynamical evolution as it turns out from our numerical results in the gaussian approximation.

Let us begin with the precise form of the evolution equations for the field background and the quantum mode widths, as described in sections III B (cfr eq. (3.29)).

$$\left[ \frac{d^2}{dt^2} + (M^2 - 2\lambda\phi^2) \right] \phi = 0 , \quad \left[ \frac{d^2}{dt^2} + k_n^2 + M^2 \right] \sigma_n - \frac{1}{4\sigma_n^3} = 0 \quad (5.1)$$

where the index  $n$  labels the discrete set of values used to perform the sum (finite volume) or the integral (infinite volume) over momenta in the quantum back-reaction  $\Sigma$ , while  $M^2(t)$  is defined by the eq. (3.30) in case of unbroken symmetry and by eq. (3.31) in case of broken symmetry. The back-reaction  $\Sigma$  reads, in the notations of this appendix

$$\Sigma = \sum_{n=0}^{\mathcal{N}} g_n \sigma_n^2$$

where  $g_n$  is the appropriate “degeneracy” factor and  $\mathcal{N}$  is the number of modes with distinct dynamics. Technically it is simpler to treat an equivalent set of equations, which are formally linear and do not contain the singular Heisenberg term  $\propto \sigma_n^{-3}$ . This is done by introducing the complex mode amplitudes  $z_n = \sigma_n \exp(i\theta_n)$ , where the phases  $\theta_n$  satisfy  $\sigma_n^2 \dot{\theta}_n = 1$ . Then we find a discrete version of the equations studied for instance in ref [1], namely

$$\left[ \frac{d^2}{dt^2} + k_n^2 + M^2 \right] z_n = 0 , \quad \Sigma = \frac{1}{L^D} \sum_{n=0}^{\mathcal{N}} g_n |z_n|^2 \quad (5.2)$$

subject to the Wronskian condition

$$z_n \dot{\bar{z}}_n - \bar{z}_n \dot{z}_n = -i$$

One realizes that the Heisenberg term in  $\sigma_n$  corresponds to the centrifugal potential for the motion in the complex plane of  $z_n$ . Looking at the figs. 2 or 9, we can see that the motions of the quantum modes correspond qualitatively to orbits with very large eccentricities. In fact, there are instants in which  $\sigma_n$  is very little and the angular velocity  $\dot{\theta}_n$  is very large. This is the technical reason for preferring the equations in the form (5.2).

To solve these evolution equations, we have to choose suitable initial conditions respecting the Wronskian condition. In case of unbroken symmetry, the requirement of minimum energy for the quantum fluctuations leads to the massive particle spectrum:

$$z_n(0) = \frac{1}{\sqrt{2\Omega_n}} \quad \frac{dz_n}{dt}(0) = i\sqrt{\frac{\Omega_n}{2}}$$

where  $\Omega_n = \sqrt{k_n^2 + M^2(0)}$  and the initial squared effective mass  $M^2(t = 0)$ , has to be determined self-consistently, by means of its definition (3.30).

In case of broken symmetry, the gap equation is a viable mean for fixing the initial conditions only when  $\phi$  lies outside the spinodal region [cfr. eq (3.35)]; otherwise, the gap equation does not admit a positive solution for the squared effective mass and we cannot minimize the energy of the fluctuations. Following the discussion presented in III B, one possible choice is to set  $\sigma_k^2 = \frac{1}{2\sqrt{k^2 + |M^2|}}$  for  $k^2 < |M^2|$  and then solve the corresponding gap equation (3.36). We will call this choice the “flipped” initial condition. An other acceptable choice would be to solve the gap equation, setting a massless spectrum for all the spinodal modes but the zero mode, which is started from an arbitrary, albeit microscopic, value. This choice will be called the “massless” initial condition.

Before passing to discuss the influence of different initial conditions on the results, let us present the asymptotic behavior we find when we choose the flipped initial condition. In Fig. 13 we have plotted the asymptotic values of the mean field versus the initial values, for  $\lambda = 0.1$ . All dimensionful quantities are expressed in terms of the suitable power of the equilibrium mass  $m$ . For example, the vev of the field is equal to  $\sqrt{5}$  in these units. First of all, consider the initial values for the condensate far enough from the top of the potential hill, say between  $\bar{\phi}(t=0) = 0.88$  and  $\bar{\phi}(t=0) = 2.64$ . In that region the crosses seem to follow a smooth curve, that has its maximum exactly at  $\bar{\phi}_\infty = \sqrt{5}$  (the point of stable equilibrium). When we start from an initial condition smaller than  $\bar{\phi}(t=0) = 0.88$ , the asymptotic value  $\bar{\phi}_\infty$  is not guaranteed to be positive anymore. On the contrary, it is possible to choose the initial condition in such a way that the condensate will oscillate between positive and negative values for a while, before settling around an asymptotic value near either one or the other minimum, as fig 11 clearly shows. Fig.s 14, 15 and 16 helps to understand this behavior by consideration on the energy balance. Both the evolutions are such that the classical energy, defined as  $\lambda(\bar{\phi}^2 - v^2)/4$ , is not a monotonically decreasing function of time. Indeed, energy is continuously exchanged between the classical degree of freedom and the quantum fluctuations bath, in both directions. However, the two rates of energy exchange are not exactly the same and an effective dissipation of classical energy on average can be seen, at long time at least. Of course, this is not the case for the initial transient part of the evolution starting from the initial condition  $\bar{\phi}(t=0) = 0.08$ ; there, the condensate absorbs energy (on average) from the quantum fluctuations, being able to go beyond the top of the potential hill, towards the negative minimum. This happens because in case of broken symmetry, the minimization of the fluctuation energy, within microscopic gaussian states, is not possible for initial conditions in the spinodal region [cfr. the discussion about the gap equation (3.35) in section III B]. After a number of oscillations, the energy starts to flow from the condensate to the quantum bath again (on the average), constraining the condensate to oscillate around a value close to one of the two minima. If we look at fig. 13 again, we can find positive asymptotic values as well as negative ones, without a definite pattern, in the whole interval  $[0.01, 0.8]$ . If we start with  $0 < \bar{\phi}(t=0) < 0.01$  we have the slow rolling down, already described in section IV and the mean field oscillates around a positive value from the beginning, never reaching negative values. A further note is worth being added here. During the phase of slow rolling down, the evolution is very similar to a symmetric evolution starting from  $\bar{\phi}(t=0) = 0$ ; in that case, the dissipation mechanism works through the emission of (quasi-)massless particles and it is very efficient because it has not any perturbative threshold. If the field stays in this slow rolling down phase for a time long enough, it will not be able to absorb the sufficient energy to pass to the other side ever

again and it will be confined in the positive valley for ever. Evidently, when  $\bar{\phi}(t=0) > 0.01$  this dissipative process might not be so efficient to prevent the mean field from sampling also the other valley.

Which one of the two valleys will be chosen by the condensate is a matter of initial conditions and it is very dependent from the energy stored in the initial state, as is shown in fig. 12, where two evolutions are compared, starting from the same value for the condensate, but with the two initial conditions, “flipped” and massless, for the quantum fluctuations.

## VI. CONCLUSIONS AND PERSPECTIVES

In this work we have extended the standard time dependent Hartree-Fock approximation [13] for the  $\phi^4$  QFT, to include some non-gaussian features of the complete theory. We have presented a rather detailed study of the dynamical evolution out of equilibrium, in finite volume (a cubic box of size  $L$  in 3D), as well as in infinite volume. For comparison, we have also analyzed some static characteristics of the theory both in unbroken and broken symmetry phases.

By means of a proper substitution of the bare coupling constant with the renormalized coupling constant (fully justified by diagrammatic consideration), we have been able to obtain equations of motion completely independent of the ultraviolet cut-off (apart from a slight dependence on inverse powers, that is, however, ineluctable because of the Landau pole). We have described in detail the shape of the ground state, showing how a broken symmetry scenario can be recovered from the quantum mechanical model, when the volume diverges.

Moreover, we have shown that, within this slightly enlarged tdHF approach that allows for non-gaussian wavefunctions, one might recover the usual gaussian HF approximation in a more controlled way. In fact, studying the late time dynamics, we have confirmed the presence of a time scale  $\tau_L$ , proportional to the linear size  $L$  of the box, at which the evolution ceases to be similar to the infinite volume one. At the same time, the low-lying modes amplitudes have grown to order  $L$ . The same phenomenon has been observed in the  $O(N)$  model [8]. Looking at this result in the framework of our extended tdHF approximation, one realizes that the growth of long-wavelength fluctuations to order  $L$  in fact undermines the self-consistency of the gaussian HF itself. In fact, in our tdHF approach the initial gaussian wavefunctions are allowed to evolve into non-gaussian forms, but they simply do not do it in a macroscopic way, within a further harmonic approximation for the evolution, so that in the infinite-volume limit they are indistinguishable from gaussians at all times. But when  $M^2$  is on average not or order  $L^0$ , but much less, as it happens for suitable initial conditions, infrared modes of order  $L$  will be dominated by the quartic term in our Schroedinger equations (3.5), showing a possible internal inconsistency of the gaussians approximation.

An other manifestation of the weakness of the HF scheme is the curious “stopping at the spinodal line” of the width of the gaussian quantum fluctuations, when the initial configuration does not break the symmetry. This does not happen in the large  $N$  approach because of different coupling of transverse mode (the only ones that survive in the  $N \rightarrow \infty$  limit) with respect to the longitudinal modes of the  $N = 1$  case in the HF approach.

We have also described the non-trivial phenomenology of the infinite-volume late-time evolution in the gaussian approximation, showing how the dynamical Maxwell construction differs from the  $N = \infty$  case. In fact, we have observed the presence of an unstable interval, contained in the static flat region which is forbidden as attractor of the asymptotic evolution. This region corresponds, more or less, to the spinodal region of the classical potential, with the obvious exception of the origin. In particular, we have found that the energy flux between the classical degree of freedom and the bath of quantum fluctuations is quite complex and not monotonous. In other words, since we start from initial conditions where the fluctuation energy is not minimal, there are special situations where enough energy is transferred from the bath to the condensate, pushing it beyond the top of the potential hill.

Clearly further study, both analytical and numerical, is needed in our tdHF approach to better understand the dynamical evolution of quantum fluctuations in the broken symmetry phase coupled to the condensate. An interesting direction is the investigation of the case of finite  $N$ , in order to interpolate smoothly the results for  $N = 1$  to those of the  $1/N$  approach. It should be noted, in fact, that the theory with a single scalar field contains only the longitudinal mode (by definition), while only the transverse modes are relevant in the large  $N$  limit. Hence a better understanding of the coupling between longitudinal and transverse modes is necessary.

In this direction, another relevant point is whether the Goldstone theorem is respected in the HF approximation [20]. It would be interesting also to study the dynamical realization of the Goldstone paradigm, namely the asymptotic vanishing of the effective mass in the broken symmetry phases, in different models; this issue needs further study in the  $2D$  case [2], where it is known that the Goldstone theorem is not valid.

## VII. ACKNOWLEDGEMENTS

C. D. thanks D. Boyanovsky, H. de Vega, R. Holman and M. Simionato for very interesting discussions. C. D. and E. M. thank MURST and INFN for financial support.

## FIGURES

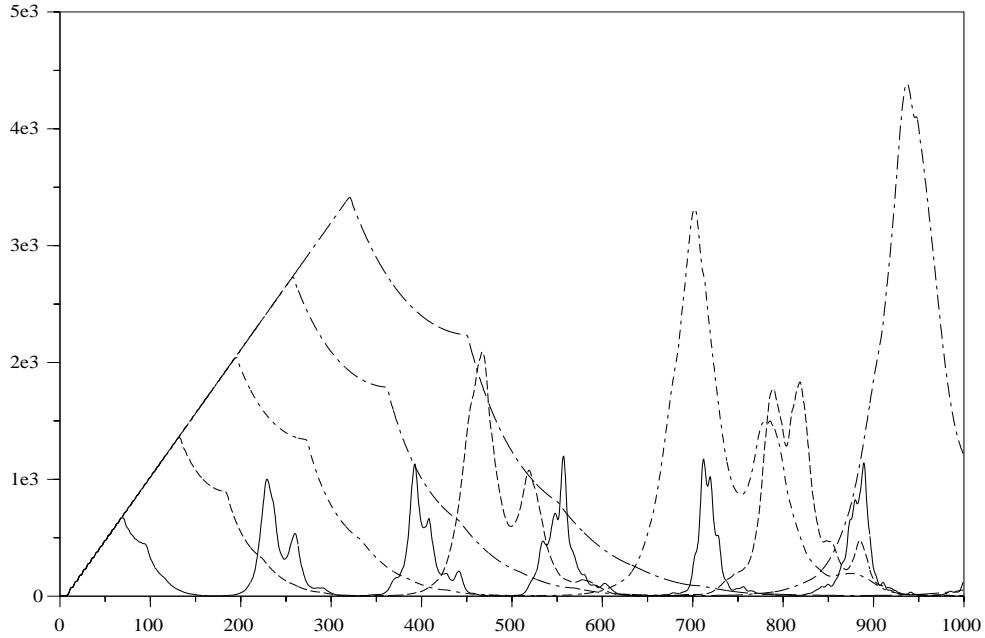


FIG. 1. Zero-mode amplitude evolution for different values of the size  $\frac{L}{2\pi} = 20, 40, 60, 80, 100$ , for  $\lambda = 0.1$  and broken symmetry, with  $\bar{\phi} = 0$ .

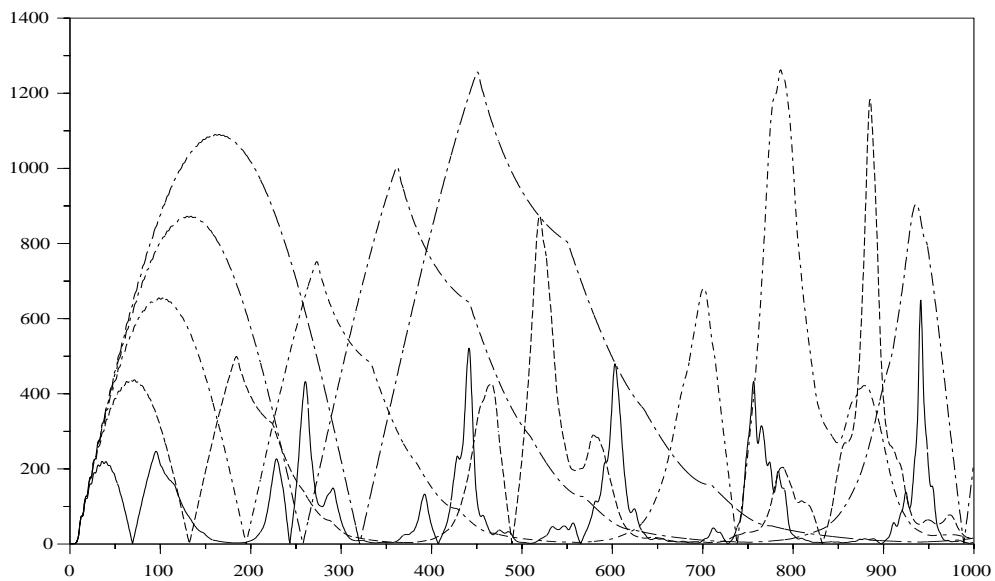


FIG. 2. Next-to-zero mode ( $k = 2\pi/L$ ) amplitude evolution for different values of the size  $L = 20, 40, 60, 80, 100$ , for  $\lambda = 0.1$  and broken symmetry, with  $\bar{\phi} = 0$ .

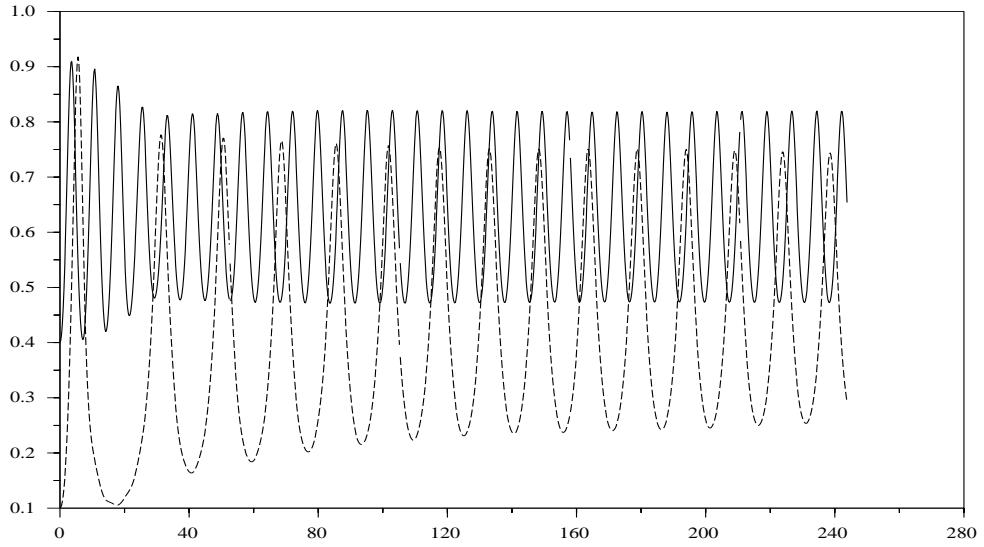


FIG. 3. *Evolution of the background for two different initial conditions within the spinodal interval, in the tdHF approximation, for  $\lambda = 1$ :  $\bar{\phi}(t = 0) = 0.1$  (dotted line) and  $\bar{\phi}(t = 0) = 0.4$  (solid line).*

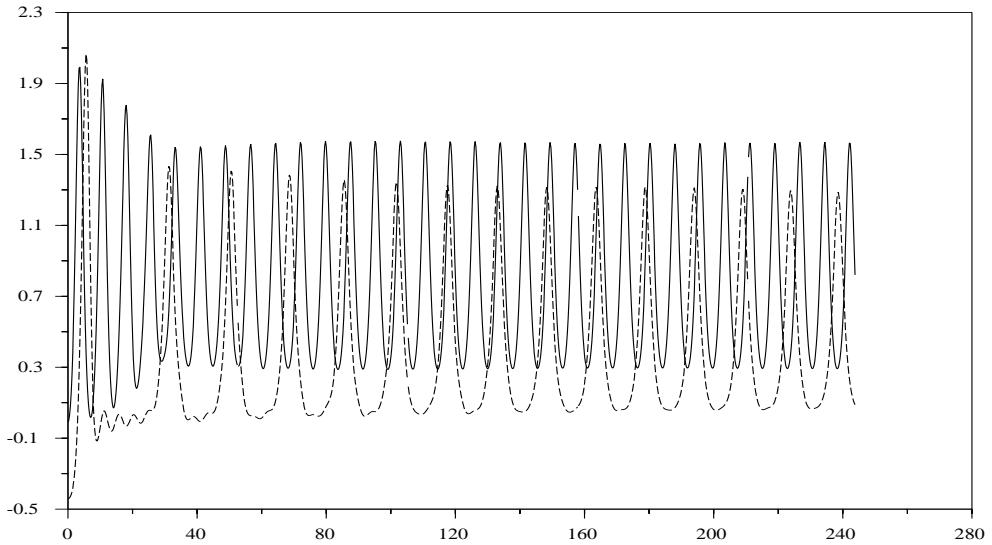


FIG. 4. *Evolution of  $M^2$  for the two initial conditions of fig. 3.*

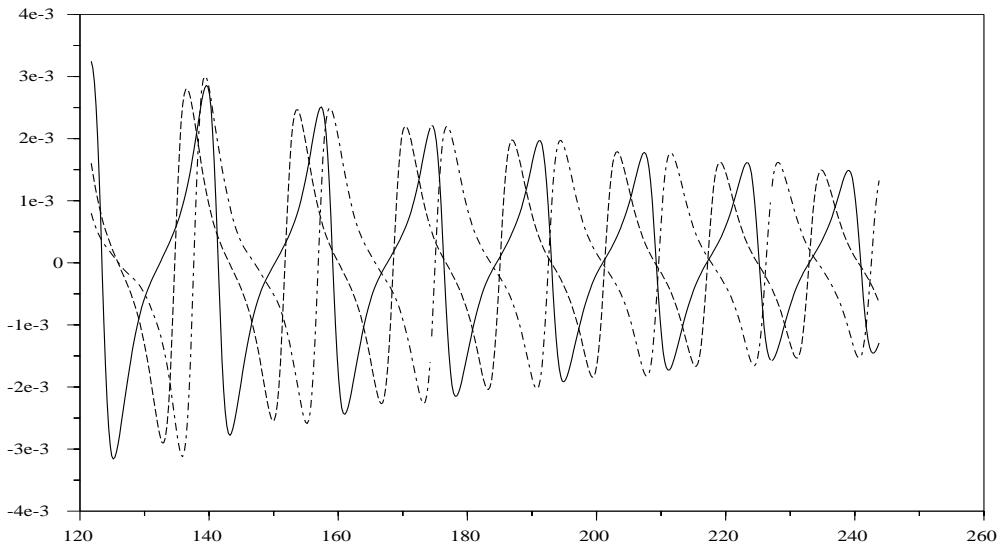


FIG. 5. The average force  $f$ , defined as  $\bar{f} = \int^T f(t)dt/T$ , plotted vs.  $T$ , for  $\lambda = 0.1$  and  $\bar{\phi} = 10^{-2}$  (solid line),  $\bar{\phi} = 10^{-3}$  (dashed line) and  $\bar{\phi} = 10^{-4}$  (dotted-dashed line).

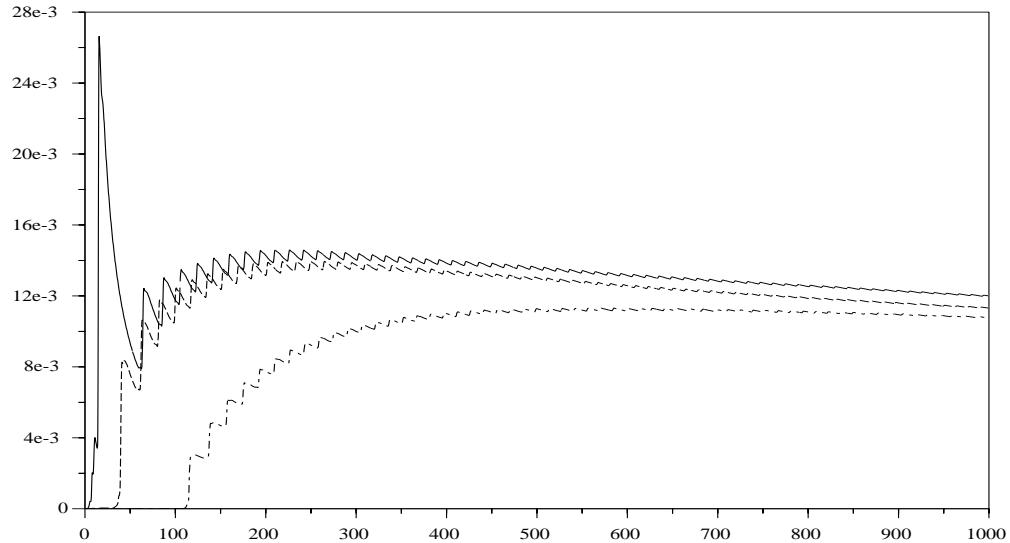


FIG. 6. The mean squared fluctuations of the force  $f$ , defined as  $\int^T (f(t) - \bar{f})^2 dt/T$ , plotted vs.  $T$ , for the three initial conditions of fig. 5.

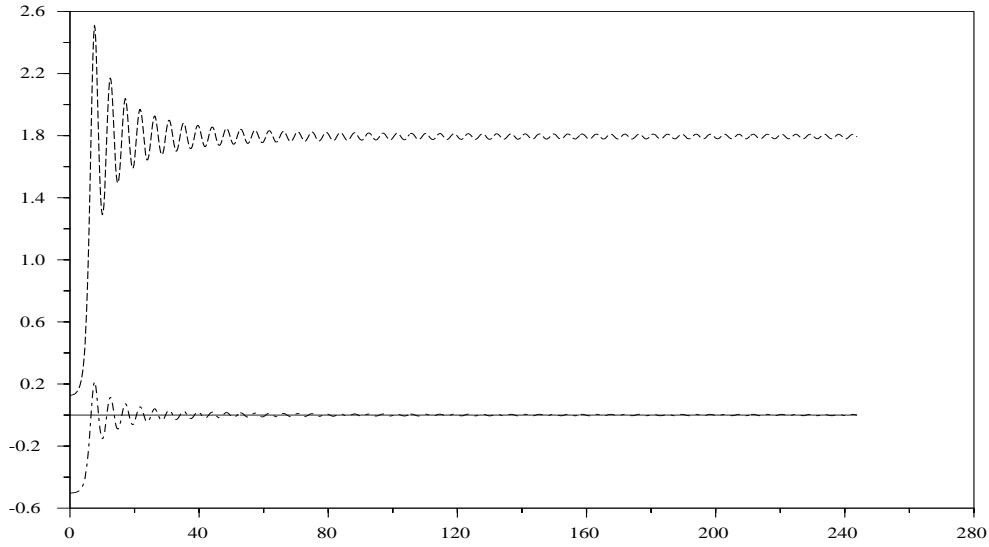


FIG. 7. The evolution of the mean value (solid line), the quantum back-reaction  $\Sigma$  (dashed line) and the squared effective mass  $M^2$  (dotted-dashed line), for  $\bar{\phi} = 0$  at  $t = 0$ .

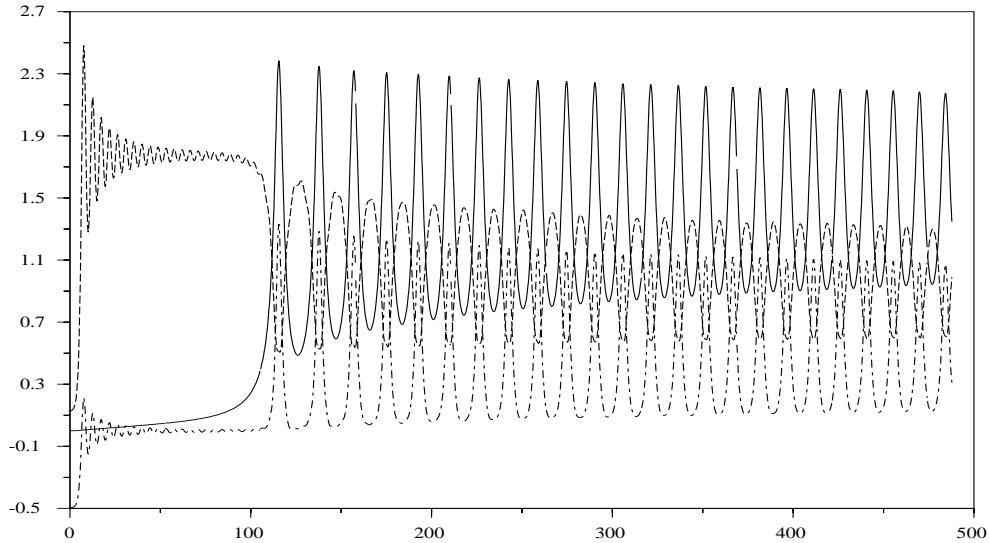


FIG. 8. The evolution of the mean value (solid line), the quantum back-reaction  $\Sigma$  (dashed line) and the squared effective mass  $M^2$  (dotted-dashed line), for  $\bar{\phi} = 10^{-4}$  at  $t = 0$ , and  $\lambda = 0.1$ . The field rolls down very slowly at the beginning.

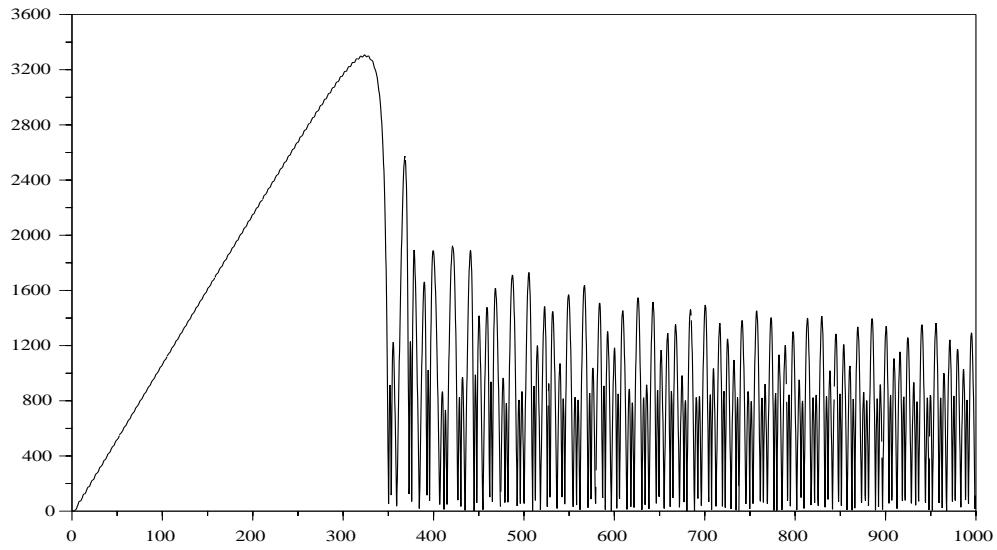


FIG. 9. *Evolution of the amplitude of the zero mode for  $\lambda = 0.1$  and  $\bar{\phi} = 10^{-5}$ .*

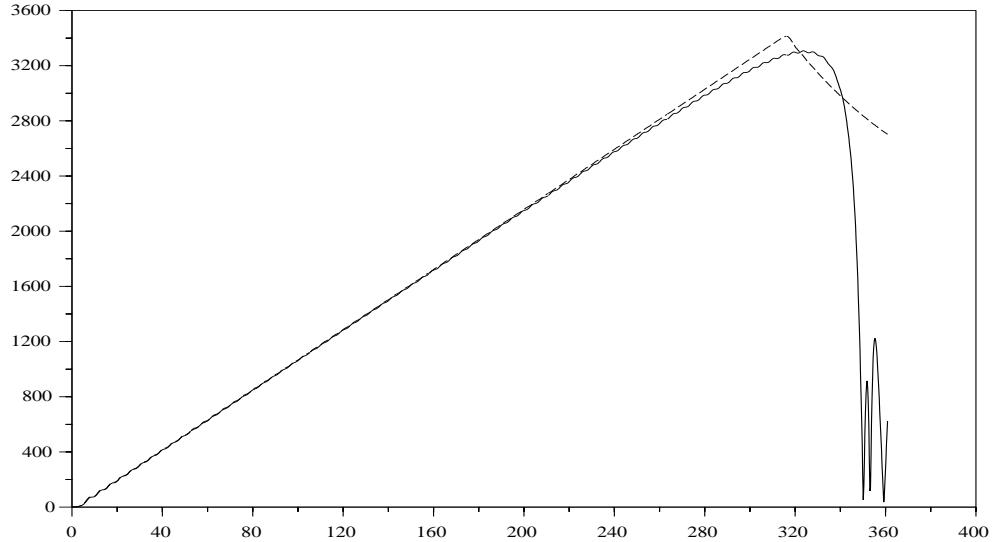


FIG. 10. *Comparison between the evolutions of the zero mode amplitude in the following two situations: the dashed line corresponds to a finite volume simulation with  $L/2\pi = 100$  and  $\bar{\phi} = 0$ , while the solid line refers to the infinite volume evolution, with  $\bar{\phi} = 10^{-5}$ . Both correspond to  $\lambda = 0.1$ .*

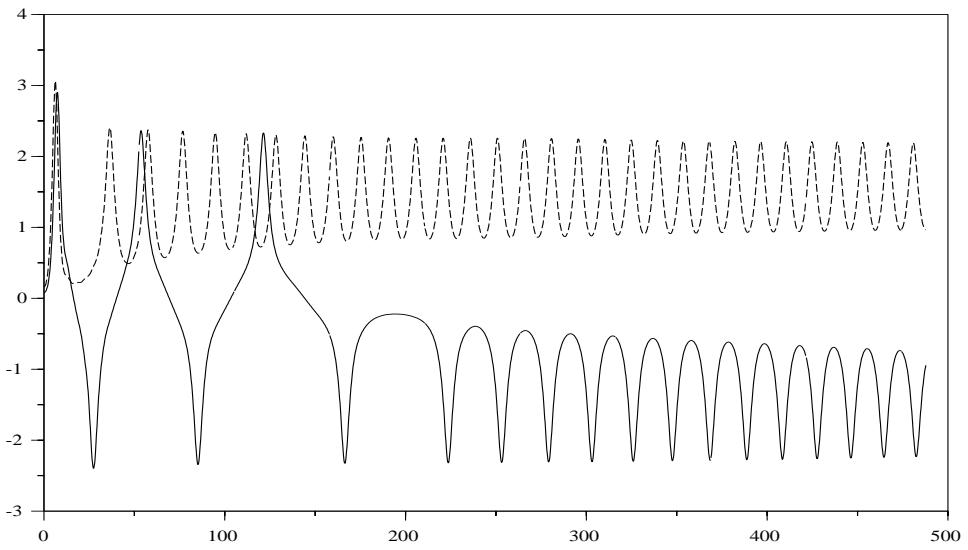


FIG. 11. Evolution of the mean value  $\bar{\phi}$  for  $\lambda = 0.1$  and for two different initial conditions:  $\bar{\phi} = 0.08$  (solid line) and  $\bar{\phi} = 0.16$  (dashed line), with the “flipped” choice for the spinodal modes.

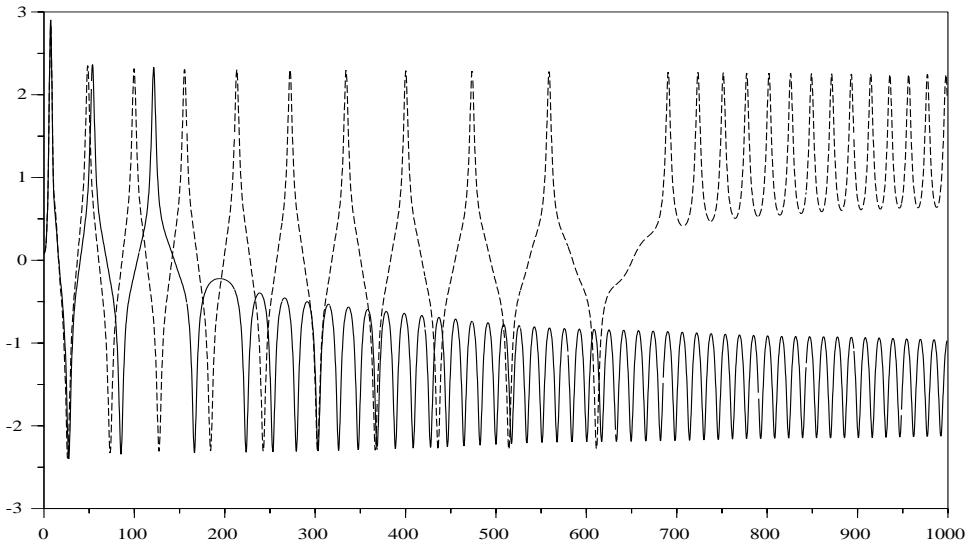


FIG. 12. Evolution of the mean value  $\bar{\phi}$  for  $\lambda = 0.1$ , with  $\bar{\phi}(t = 0) = 0.08$ , and two different initial conditions for the quantum spinodal modes, “flipped” (solid line) and massless (dashed line).

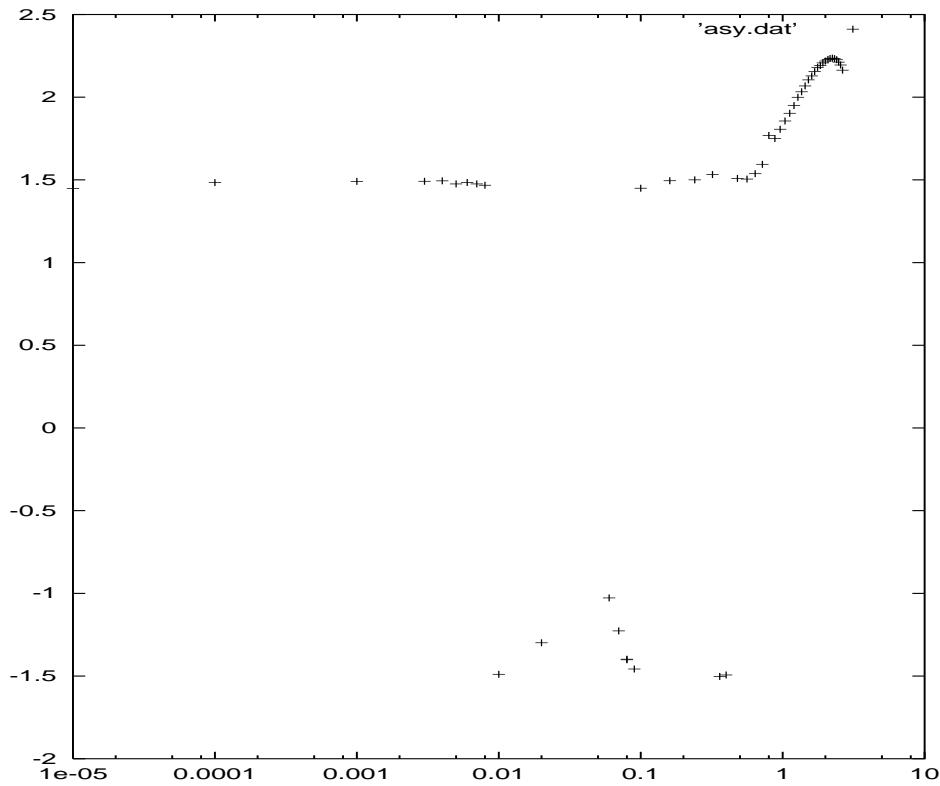


FIG. 13. Asymptotic values of the mean field  $\bar{\phi}$ , plotted vs. initial values  $\bar{\phi}(t = 0)$ , for  $\lambda = 0.1$ .

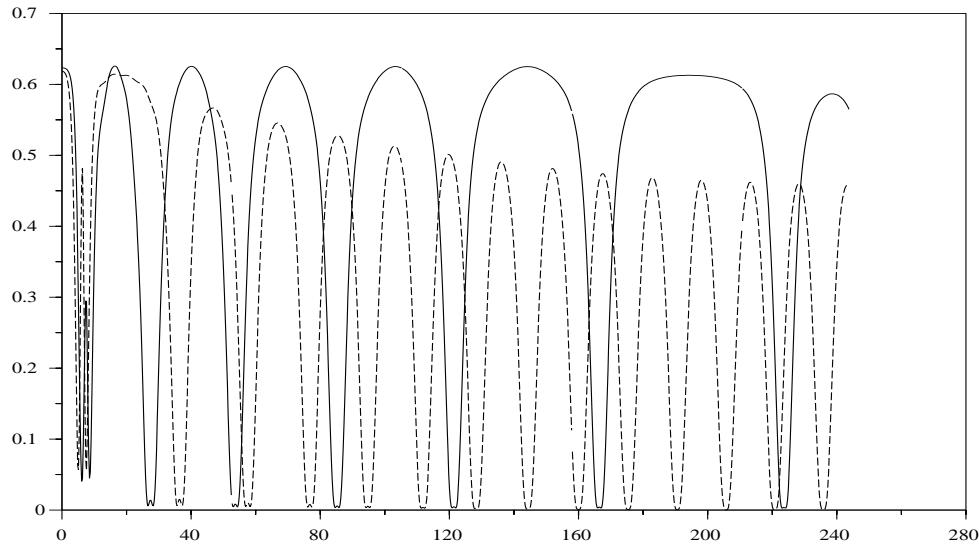


FIG. 14. Comparison between the classical energies for the two initial conditions of Fig 11.

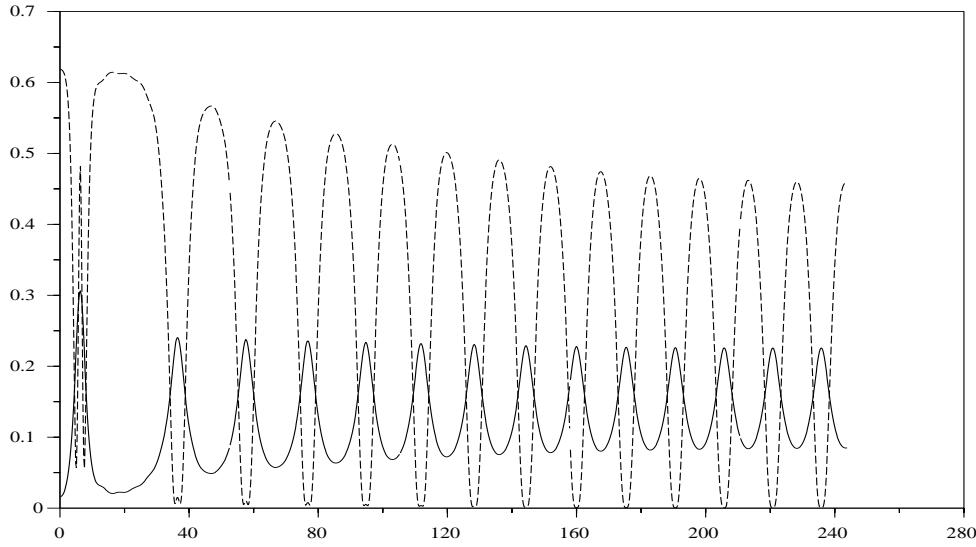


FIG. 15. *Evolution of the condensate  $\bar{\phi}$  (solid line) and of the corresponding classical energy (dashed line), for  $\bar{\phi}(t = 0) = 0.16$  and  $\lambda = 0.1$  (cfr. Fig.s 14 and 11).*

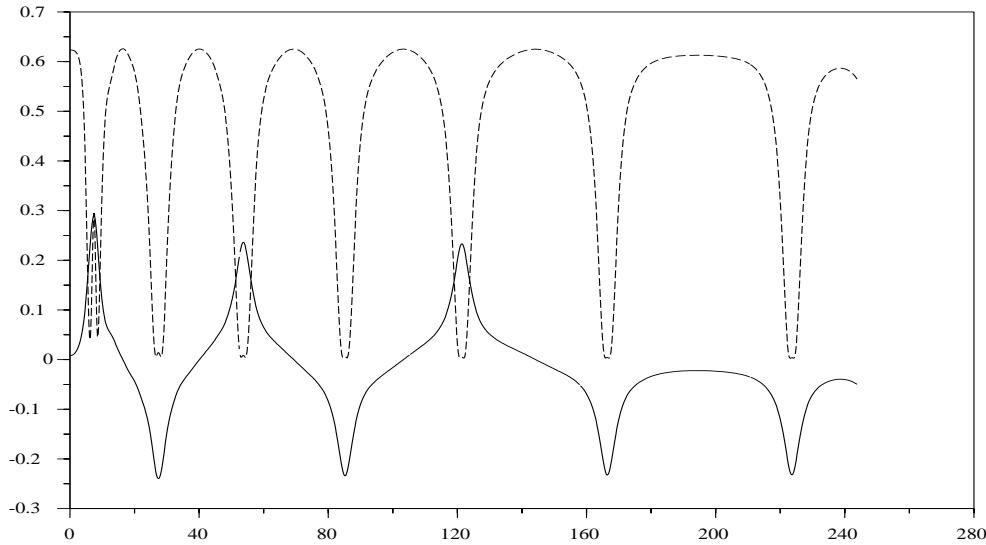


FIG. 16. *Evolution of the condensate  $\bar{\phi}$  (solid line) and of the corresponding classical energy (dashed line), for  $\bar{\phi}(t = 0) = 0.08$  and  $\lambda = 0.1$  (cfr. Fig.s 14 and 11).*

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